Math/Stat 251 Fall 2015
Summary of Lecture from October 5, 2015 and October 7, 2015

Until now, all of our examples have involved sample spaces with a finite number of outcomes. We would now like to consider the case where the sample space contains a continuum of outcomes; that is, we want to have the sample space $S = \mathbb{R}$. One way to define probabilities for subsets of $\mathbb{R}$ is through the use of an auxiliary function known as a *probability density function*. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has the properties that

(a) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and

(b) $\displaystyle\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

If we define

$$\mathbf{P}(A) = \int_A f(x)\,\mathrm{d}x$$

for any event $A \subset \mathbb{R}$, then this defines a probability. Note that

$$\mathbf{P}(\emptyset) = \int_\emptyset f(x)\,\mathrm{d}x = 0, \quad \mathbf{P}(S) = \mathbf{P}(\mathbb{R}) = \int_\mathbb{R} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1,$$

and if A and B are disjoint, then

$$\mathbf{P}(A \cup B) = \int_{A \cup B} f(x)\,\mathrm{d}x = \int_A f(x)\,\mathrm{d}x + \int_B f(x)\,\mathrm{d}x = \mathbf{P}(A) + \mathbf{P}(B).$$

Furthermore, since $f \geq 0$, we conclude that

$$\mathbf{P}(A) = \int_A f(x)\,\mathrm{d}x \geq 0$$

and that since $A \subseteq \mathbb{R}$, we have

$$\mathbf{P}(A) = \int_A f(x)\,\mathrm{d}x \leq \int_\mathbb{R} f(x)\,\mathrm{d}x = 1.$$

That is, $0 \leq \mathbf{P}(A) \leq 1$ so that $\mathbf{P}$ is a legitimate probability.

The following six examples of density functions are of particular importance for this course. We will be using them continually.

Example 1. Suppose that $\lambda > 0$ and let

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

This is the exponential density with parameter λ.

Example 2. Suppose that $-\infty < a < b < \infty$ and let

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

This is the uniform density on the interval $[a, b]$.

Example 3. Suppose that $-\infty < \theta < \infty$ and let

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x-\theta)^2}$$

for $-\infty < x < \infty$. This is the Cauchy density with parameter θ.

Example 4. Suppose that $\lambda > 0$, $\alpha > 0$, and let

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Here, Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, \mathrm{d}x.$$

This is the gamma density with parameters λ and α.

Example 5. Suppose that $-\infty < \mu < \infty$, $\sigma > 0$, and let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

for $-\infty < x < \infty$. This is the normal (or Gaussian) density with mean μ and variance σ^2.

Example 6. Suppose that $a > 0$, $b > 0$, and let

$$f(x) = \begin{cases} \dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

This is the beta density with parameters a and b.

In order to verify that these six functions are legitimate density functions, we need to verify that each is non-negative and integrates to 1. Clearly all six are non-negative. As for the fact that each integrates to 1, the first three can be verified by direct integration.

Example 1.

$$\int_{-\infty}^\infty f(x) \, \mathrm{d}x = \int_{-\infty}^0 f(x) \, \mathrm{d}x + \int_0^\infty f(x) \, \mathrm{d}x = \int_{-\infty}^0 0 \, \mathrm{d}x + \int_0^\infty \lambda e^{-\lambda x} \, \mathrm{d}x = -e^{-\lambda x} \Big|_0^\infty = 1.$$

Example 2.

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_{a}^{b} f(x)\,\mathrm{d}x + \int_{b}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} 0\,\mathrm{d}x + \int_{a}^{b} \frac{\mathrm{d}x}{b-a} + \int_{b}^{\infty} 0\,\mathrm{d}x$$

$$= \frac{x}{b-a}\bigg|_{a}^{b} = 1.$$

Example 3.

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\pi}\cdot\frac{1}{1+(x-\theta)^2}\,\mathrm{d}x = \frac{1}{\pi}\arctan(x-\theta)\bigg|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1.$$

Example 4. The fact that the gamma density integrates to 1 is a consequence of the fact that the gamma function is well-defined.

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{0}^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\,\mathrm{d}x = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_{0}^{\infty} \frac{u^{\alpha-1}}{\lambda^{\alpha-1}}e^{-u}\,\frac{\mathrm{d}u}{\lambda} = \frac{1}{\Gamma(\alpha)}\int_{0}^{\infty} u^{\alpha-1}e^{-u}\,\mathrm{d}u$$

$$= \frac{1}{\Gamma(\alpha)}\cdot\Gamma(\alpha)$$

$$= 1$$

Multivariable calculus is required to prove that the remaining two examples actually define legitimate density functions. Since Math 213 is not a prerequisite for this class, we will not prove these facts.