Math/Stat 251 Fall 2015
Solutions to Assignment #3

1. Let R_j, $j = 1, 2, \ldots, 50$, be the event that the jth box of Raisin Bran contains 2 scoops of raisins. We are told that $\mathbf{P}\{R_j\} = 0.93$ for each j and that the events $R_1, R_2, \ldots, R_{50}$ are independent.

(a) The probability that all of Danny's 50 packages contain two scoops of raisins is

$$\mathbf{P}\{R_1 \cap R_2 \cap \cdots \cap R_{50}\} = \mathbf{P}\{R_1\}\,\mathbf{P}\{R_2\} \cdots \mathbf{P}\{R_{50}\} = (0.93)^{50}.$$

(b) Notice that

$$\begin{aligned}&\{\text{at least one of Danny's 50 packages does not contain two scoops}\}^c\\ &\qquad = \{\text{none of Danny's 50 packages do not contain two scoops}\}\\ &\qquad = \{\text{all of Danny's 50 packages contain two scoops}\}\end{aligned}$$

This implies that

$$\begin{aligned}&\mathbf{P}\{\text{at least one of Danny's 50 packages does not contain two scoops}\}\\ &\qquad = 1 - \mathbf{P}\{\text{all of Danny's 50 packages contain two scoops}\}\\ &\qquad = 1 - (0.93)^{50}.\end{aligned}$$

Symbolically, the event {at least one of Danny's 50 packages does not contain two scoops} can be written as $R_1^c \cup R_2^c \cup \cdots \cup R_{50}^c$ and so

$$\begin{aligned}\mathbf{P}\{R_1^c \cup R_2^c \cup \cdots \cup R_{50}^c\} = 1 - \mathbf{P}\{(R_1^c \cup R_2^c \cup \cdots \cup R_{50}^c)^c\} &= 1 - \mathbf{P}\{R_1 \cap R_2 \cap \cdots \cap R_{50}\}\\ &= 1 - (0.93)^{50}\end{aligned}$$

using the fact that $(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B$. (This is sometimes called De Morgan's law.)

(c) Notice that there are several ways in which exactly one package that does not contain two scoops of raisins could happen. It might be that only the first box does not contain two scoops, or it might be that only the second box does not contain two scoops, etc. Thus,

$$\begin{aligned}&\mathbf{P}\{\text{exactly one of Danny's 50 packages does not contain two scoops}\}\\ &\quad = \mathbf{P}\{(R_1^c \cap R_2 \cap \cdots \cap R_{50}) \cup (R_1 \cap R_2^c \cap R_3 \cap \cdots \cap R_{50}) \cup \cdots \cup (R_1 \cap R_2 \cap \cdots \cap R_{50}^c)\}\\ &\quad = \mathbf{P}\{R_1^c \cap R_2 \cap \cdots \cap R_{50}\} + \mathbf{P}\{R_1 \cap R_2^c \cap R_3 \cap \cdots \cap R_{50}\} + \cdots + \mathbf{P}\{R_1 \cap R_2 \cap \cdots \cap R_{50}^c\}\\ &\quad = (0.07)(0.93)\cdots(0.93) + (0.93)(0.07)(0.93)\cdots(0.93) + \cdots + (0.93)\cdots(0.93)(0.07)\\ &\quad = (0.07)(0.93)^{49} + (0.07)(0.93)^{49} + \cdots + (0.07)(0.93)^{49}\\ &\quad = 50(0.07)(0.93)^{49}.\end{aligned}$$

2. **(a)** Let A_j be the event that the jth song is by Arcade Fire. We are interested in computing $\mathbf{P}\{A_1 \cap A_2 \cap A_3\}$. The events A_1, A_2, and A_3 are NOT independent since we are not allowed to repeat a song in the playlist. Intuitively, there is a 3/20 chance that the first song is by Arcade Fire. Given that the first song is by Arcade Fire, there is a 2/19 chance that the second song will be by Arcade Fire, and finally, given that the first two songs are by Arcade Fire, the probability that the third will also be by them is 1/18. Thus, the required probability is

$$\mathbf{P}\{A_1 \cap A_2 \cap A_3\} = \mathbf{P}\{A_1\}\,\mathbf{P}\{A_2 \mid A_1\}\,\mathbf{P}\{A_3 \mid A_1 \cap A_2\} = \frac{3}{20} \cdot \frac{2}{19} \cdot \frac{1}{18} = \frac{1}{1140}.$$

(b) This problem is solved exactly the same way as **(a)**. Thus, if H_j is the event that the jth song is by Hawksley Workman, then the required probability is

$$\mathbf{P}\{H_1 \cap H_2 \cap H_3\} = \mathbf{P}\{H_1\}\,\mathbf{P}\{H_2 \mid H_1\}\,\mathbf{P}\{H_3 \mid H_1 \cap H_2\} = \frac{6}{20} \cdot \frac{5}{19} \cdot \frac{4}{18} = \frac{1}{57}.$$

(c) In order to solve this problem we will list all of the different ways that the first three songs can be by different artists. To begin, we select the three different artists (in any order). The possibilities are

{Hawksley Workman, Stars, Arcade Fire}, {Hawksley Workman, Stars, Serena Ryder},

{Hawksley Workman, Arcade Fire, Serena Ryder}, {Stars, Arcade Fire, Serena Ryder}.

Thus, if H, A, T, and R are the events that one of the first three songs are by Hawksley Workman, Arcade Fire, Stars, and Serena Ryder, respectively, then

$$\begin{aligned}
&\mathbf{P}\{\text{1st three songs are by different artists}\}\\
&\qquad = \mathbf{P}\{(H \cap T \cap A) \cup (H \cap T \cap R) \cup (H \cap A \cap R) \cup (T \cap A \cap R)\}\\
&\qquad = \mathbf{P}\{H \cap T \cap A\} + \mathbf{P}\{H \cap T \cap R\} + \mathbf{P}\{H \cap A \cap R\} + \mathbf{P}\{T \cap A \cap R\}.
\end{aligned}$$

However, within each group, there are six different orderings. That is, if the group is $\{a, b, c\}$, then the orderings are abc, acb, bac, bca, cab, cba. This means that in order to compute $\mathbf{P}\{H \cap T \cap A\}$, we must consider the six different ways that the first three songs can be by Hawksley Workman, Stars, and Arcade Fire. Thus, if H_j, T_j, and A_j represent the events that the jth song is by Hawksley Workman, Stars, and Arcade Fire, respectively, then

$$\begin{aligned}
&\mathbf{P}\{H \cap T \cap A\}\\
&\quad = \mathbf{P}\{(H_1 \cap T_2 \cap A_3) \cup (H_1 \cap A_2 \cap T_3) \cup (T_1 \cap H_2 \cap A_3) \cup (T_1 \cap A_2 \cap H_3)\\
&\qquad\qquad \cup (A_1 \cap T_2 \cap H_3) \cup (A_1 \cap H_2 \cap T_3)\}\\
&\quad = \mathbf{P}\{H_1 \cap T_2 \cap A_3\} + \mathbf{P}\{H_1 \cap A_2 \cap T_3\} + \mathbf{P}\{T_1 \cap H_2 \cap A_3\} + \mathbf{P}\{T_1 \cap A_2 \cap H_3\}\\
&\qquad\qquad + \mathbf{P}\{A_1 \cap T_2 \cap H_3\} + \mathbf{P}\{A_1 \cap H_2 \cap T_3\}\\
&\quad = \frac{6}{20} \cdot \frac{4}{19} \cdot \frac{3}{18} + \frac{6}{20} \cdot \frac{3}{19} \cdot \frac{4}{18} + \frac{4}{20} \cdot \frac{6}{19} \cdot \frac{3}{18} + \frac{4}{20} \cdot \frac{3}{19} \cdot \frac{6}{18} + \frac{3}{20} \cdot \frac{4}{19} \cdot \frac{6}{18} + \frac{3}{20} \cdot \frac{6}{19} \cdot \frac{4}{18}\\
&\quad = 6 \cdot \frac{3 \cdot 4 \cdot 6}{20 \cdot 19 \cdot 18}.
\end{aligned}$$

Similarly,

$$\mathbf{P}\{H \cap T \cap R\} = 6 \cdot \frac{6 \cdot 4 \cdot 7}{20 \cdot 19 \cdot 18}, \quad \mathbf{P}\{H \cap A \cap R\} = 6 \cdot \frac{6 \cdot 3 \cdot 7}{20 \cdot 19 \cdot 18}, \quad \mathbf{P}\{T \cap A \cap R\} = 6 \cdot \frac{4 \cdot 3 \cdot 7}{20 \cdot 19 \cdot 18}.$$

Therefore, we conclude that

$$\begin{aligned}
&\mathbf{P}\{\text{1st three songs are by different artists}\}\\
&\quad = 6\cdot\frac{3\cdot 4\cdot 6}{20\cdot 19\cdot 18} + 6\cdot\frac{6\cdot 4\cdot 7}{20\cdot 19\cdot 18} + 6\cdot\frac{6\cdot 3\cdot 7}{20\cdot 19\cdot 18} + 6\cdot\frac{4\cdot 3\cdot 7}{20\cdot 19\cdot 18}\\
&\quad = \frac{15}{38}.
\end{aligned}$$

3. We want to compute $\mathbf{P}\{\text{Elvis was an identical twin} \mid \text{Elvis had a twin brother}\}$. Using Bayes' rule we find

$$\begin{aligned}
&\mathbf{P}\{\text{Elvis was an identical twin}|\text{Elvis had a twin brother}\}\\
&\quad = \frac{\mathbf{P}\{\text{Elvis had a twin brother}|\text{Elvis was an identical twin}\}\cdot\mathbf{P}\{\text{Elvis was an identical twin}\}}{\mathbf{P}\{\text{Elvis had a twin brother}\}}.
\end{aligned}$$

We are told that

$$\mathbf{P}\{\text{Elvis was an identical twin}\} = \frac{1}{300}.$$

Furthermore, it follows immediately that

$$\mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was an identical twin}\} = 1.$$

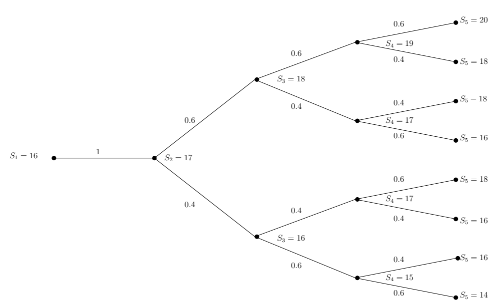
Therefore, we must calculate $\mathbf{P}\{\text{Elvis had a twin brother}\}$ using the Law of Total Probability. Thus,

$$\begin{aligned}
&\mathbf{P}\{\text{Elvis had a twin brother}\}\\
&\quad = \mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was an identical twin}\}\cdot\mathbf{P}\{\text{Elvis was an identical twin}\}\\
&\qquad + \mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was a fraternal twin}\}\cdot\mathbf{P}\{\text{Elvis was a fraternal twin}\}\\
&\qquad\quad + \mathbf{P}\{\text{Elvis had a twin brother} \mid \text{Elvis was NOT a twin}\}\cdot\mathbf{P}\{\text{Elvis was NOT a twin}\}\\
&\quad = 1\cdot\frac{1}{300} + \frac{1}{2}\cdot\frac{1}{125} + 0
\end{aligned}$$

so that

$$\mathbf{P}\{\text{Elvis was an identical twin} \mid \text{Elvis had a twin brother}\} = \frac{1\cdot\frac{1}{300}}{1\cdot\frac{1}{300}+\frac{1}{2}\cdot\frac{1}{125}} = \frac{5}{11}.$$

4. The following tree diagram shows the possible values of the stock on days 3, 4, and 5, and the corresponding probabilities. Note that tomorrow the stock has a 60% chance of doing the same thing (either increasing or decreasing) as today.



We can now answer both **(a)** and **(b)** by following the appropriate branches through the tree.

(a) We know the values of the stock on days 1 and 2. Therefore, in order to determine the probability that the stock price is \$17 on day 4, we need to use the Law of Total Probability and condition on the possible values of the stock on day 3. Thus,

$$\begin{aligned}\mathbf{P}\{S_4 = 17\} &= \mathbf{P}\{S_4 = 17 \cap S_3 = 16\} + \mathbf{P}\{S_4 = 17 \cap S_3 = 18\}\\ &= \mathbf{P}\{S_4 = 17 \,|\, S_3 = 16\}\,\mathbf{P}\{S_3 = 16\} + \mathbf{P}\{S_4 = 17 \,|\, S_3 = 18\}\,\mathbf{P}\{S_3 = 18\}\\ &= (0.4)(0.4) + (0.4)(0.6)\\ &= 0.4.\end{aligned}$$

(b) If we know that the price of the stock was \$18 on day 5, then in order to determine the probability that the stock price was \$17 on day 4, we need to consider what happened on day 3. Intuitively, we can just use the definition of conditional probability and notice that there are three paths that lead to $S_5 = 18$. Of those, there are two that have $S_4 = 17$. Thus,

$$\mathbf{P}\{S_4 = 17 \,|\, S_5 = 18\} = \frac{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4)}{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4) + (0.6)(0.6)(0.4)} = \frac{4}{7}.$$

Equivalently, we find

$$\mathbf{P}\{S_4 = 17 \,|\, S_5 = 18\} = \frac{\mathbf{P}\{S_4 = 17 \cap S_5 = 18\}}{\mathbf{P}\{S_5 = 18\}}.$$

We now use the Law of Total Probability in both the numerator and denominator. That is,

$$\begin{aligned}\mathbf{P}\{S_5 = 18\} &= \mathbf{P}\{S_5 = 18 \cap S_4 = 17\} + \mathbf{P}\{S_5 = 18 \cap S_4 = 19\}\\ &= \mathbf{P}\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 16\} + \mathbf{P}\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 18\}\\ &\qquad + \mathbf{P}\{S_5 = 18 \cap S_4 = 19 \cap S_3 = 18\}\\ &= (0.4)(0.4)(0.6) + (0.6)(0.4)(0.4) + (0.6)(0.6)(0.4)\end{aligned}$$

and

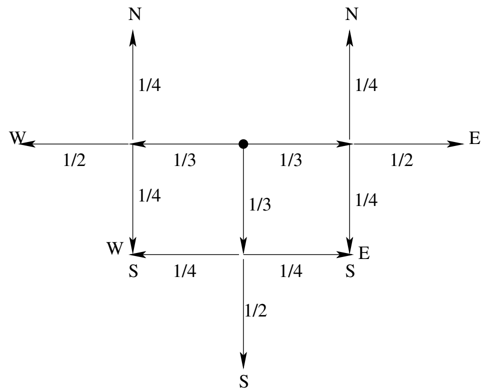
$$\begin{aligned}\mathbf{P}\{S_4 = 17 \cap S_5 = 18\} &= \mathbf{P}\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 16\} + \mathbf{P}\{S_5 = 18 \cap S_4 = 17 \cap S_3 = 18\}\\ &= (0.4)(0.4)(0.6) + (0.6)(0.4)(0.4)\end{aligned}$$

giving

$$\mathbf{P}\{S_4 = 17 \,|\, S_5 = 18\} = \frac{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4)}{(0.4)(0.4)(0.6) + (0.6)(0.4)(0.4) + (0.6)(0.6)(0.4)} = \frac{4}{7}$$

as above. Note that you could also use Bayes' Rule, but in order to do so you must account for what happens on day 3. Hence, you will be lead to precisely the same calculation.

5. In order to solve this problem, we make a *map* (i.e., a tree diagram) of the possible routes that Michael could take. Note that on the first movement, there is probability 1/3 that he moves east, south, or west, respectively. On the second movement, he changes his strategy by flipping coins as indicated in the problem.



Thus, we see that there are two possible routes that leave Michael facing west: (i) west first then straight, or (ii) south first then right. The required probability is therefore

$$\begin{aligned}
&\mathbf{P}\{\text{Michael faces west after second movement}\}\\
&\quad= \mathbf{P}\{\text{first west, second straight OR first south, second right}\}\\
&\quad= \mathbf{P}\{\text{first west, second straight}\} + \mathbf{P}\{\text{first south, second right}\}\\
&\quad= \mathbf{P}\{\text{second straight} \mid \text{first west}\}\,\mathbf{P}\{\text{first west}\} + \mathbf{P}\{\text{second right} \mid \text{first south}\}\,\mathbf{P}\{\text{first south}\}\\
&\quad= \mathbf{P}\{\text{heads}\}\cdot\frac{1}{3} + \mathbf{P}\{\text{tails, then heads}\}\cdot\frac{1}{3}\\
&\quad= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3}\\
&\quad= \frac{1}{4}.
\end{aligned}$$

6. In order for the batter to safely reach base, the batter must safely reach base on the first full count pitch, or the batter must hit a foul ball and then reach base on the second full count pitch, or the batter must hit two foul balls and then reach base safely on the third full count pitch, etc. Let F_j denote the event that the batter hits a foul ball on the jth pitch facing a full count, and let S_j denote the event that the batter safely reaches base on the jth pitch facing a full count. We are told that $\mathbf{P}\{F_j\} = 0.15$ and $\mathbf{P}\{S_j\} = 0.4$ for all j. Therefore,

$$\begin{aligned}
&\mathbf{P}\{\text{eventually safely reach base}\}\\
&\quad= \mathbf{P}\{S_1 \cup (F_1 \cap S_2) \cup (F_1 \cap F_2 \cap S_3) \cup (F_1 \cap F_2 \cap F_3 \cap S_4) \cup \cdots\}\\
&\quad= \mathbf{P}\{S_1\} + \mathbf{P}\{F_1 \cap S_2\} + \mathbf{P}\{F_1 \cap F_2 \cap S_3\} + \mathbf{P}\{F_1 \cap F_2 \cap F_3 \cap S_4\} + \cdots\\
&\quad= \mathbf{P}\{S_1\} + \mathbf{P}\{F_1\}\mathbf{P}\{S_2\} + \mathbf{P}\{F_1\}\mathbf{P}\{F_2\}\mathbf{P}\{S_3\} + \mathbf{P}\{F_1\}\mathbf{P}\{F_2\}\mathbf{P}\{F_3\}\mathbf{P}\{S_4\} + \cdots\\
&\quad= (0.4) + (0.15)(0.4) + (0.15)^2(0.4) + (0.15)^3(0.4) + \cdots\\
&\quad= (0.4)[1 + (0.15) + (0.15)^2 + (0.15)^3 + \cdots]\\
&\quad= (0.4)\cdot\frac{1}{1-0.15}\\
&\quad= \frac{40}{85}\\
&\quad= \frac{8}{17}.
\end{aligned}$$