Math 026L.04 Spring 2002
Test #4 Solutions

1.

(a) The possible values for X are: 0, 1, or 2.

(b) With probability 2/6, the player rolls 1 or 6. If that happens then with probability 1/4, the player gets 0 heads (TT), with probability 1/4 the player gets 2 heads (HH), and with probability 2/4 the player gets 1 head (TH or HT).

On the other hand, with probability 4/6, the player rolls 2, 3, 4 or 5. If that happens then with probability 1/2, the player gets 0 heads (T), and with probability 1/2 the player gets 1 head (H).

Now, we take into account whether one coin was flipped or two. The only way to win \$2 is to roll 1 or 6, and then flip two heads. Thus, $\mathbb{P}(X = 2) = \frac{2}{6} \cdot \frac{1}{4} = \frac{1}{12}$.

However, it is possible to win \$0 or \$1, by flipping either one coin or two. Thus,

$$\mathbb{P}(X = 0) = \tfrac{2}{6} \cdot \tfrac{1}{4} + \tfrac{4}{6} \cdot \tfrac{1}{2} \text{ and } \mathbb{P}(X = 1) = \tfrac{2}{6} \cdot \tfrac{2}{4} + \tfrac{4}{6} \cdot \tfrac{1}{2}.$$

Therfore the probability mass density (displayed in a table) is

$X = k$	0	1	2
$\mathbb{P}(X = k)$	$\frac{5}{12}$	$\frac{6}{12}$	$\frac{1}{12}$

(c) $\mathbb{E}(X) = 0 \cdot \mathbb{P}(X = 0) + 1 \cdot \mathbb{P}(X = 1) + 2 \cdot \mathbb{P}(X = 2) = 0 \cdot \frac{5}{12} + 1 \cdot \frac{6}{12} + 2 \cdot \frac{1}{12} = \frac{8}{12} \approx 0.67$

2.

(a)
- On the interval $[0, 2]$, $p(t)$ is clearly non-negative. In fact, t^2 is non-negative for all values of t.
- $\displaystyle\int_0^2 \frac{3}{8}t^2\, dt = \frac{1}{8}t^3\Big|_0^2 = \frac{1}{8} \cdot 8 - 0 = 1$

(b) $\displaystyle P(t) = \int_0^t p(x)\, dx = \frac{1}{8}t^3$

(c) $\displaystyle\int_0^a p(t)\, dt = \frac{1}{8}a^3$. Thus, $\displaystyle\frac{1}{8}a^3 = \frac{1}{2}$ gives $a = \sqrt[3]{4} \approx 1.587$

(d) $\displaystyle\int_0^2 t\, p(t)\, dt = \int_0^2 \frac{3}{8}t^3\, dt = \frac{3}{32}t^4\Big|_0^2 = \frac{3}{2}$

3. We begin by computing $\int t\, p(t)\, dt$.

$$\int t\, p(t)\, dt = \int \frac{2}{\pi} \frac{t}{1+t^2}\, dt = \int \frac{1}{\pi} \frac{1}{u}\, du = \frac{1}{\pi} \ln u + C = \frac{1}{\pi} \ln(1+t^2) + C$$

after substituting $u = 1 + t^2$, $du = 2\, dt$.

Thus,

$$\int_0^\infty \frac{2}{\pi} \frac{t}{1+t^2}\, dt = \lim_{b\to\infty} \int_0^b \frac{2}{\pi} \frac{t}{1+t^2}\, dt = \lim_{b\to\infty} \frac{1}{\pi} \ln(1+t^2)\Big|_0^b = \lim_{b\to\infty} \frac{1}{\pi} \ln(1+b^2) = \infty.$$

In other words, the mean $\int_0^\infty t\, p(t)\, dt$ is infinite.

4.

(a) By the FTC, $p(t) = P'(t)$. Thus,

$$p(t) = \frac{d}{dt} \frac{2}{\pi} \arcsin(\sqrt{t}) = \frac{1}{\pi} \frac{1}{\sqrt{1-t}} \frac{1}{\sqrt{t}}.$$

(b) DRAW A PICTURE TO HELP UNDERSTAND THIS SOLUTION. The fraction of students that received a score less that 0.60 is equal to the area under the *density* curve from $t = 0$ to $t = 0.60$.

But we remember that the distribution function $P(t)$ is defined as the area under the density curve from 0 to t. Thus since we want the area from 0 to 0.60, we simply compute $P(0.60)$.

Therefore, $P(0.60) = \frac{2}{\pi} \arcsin(\sqrt{0.60}) \approx 0.564$. In other words, 56.4% of students received a score less than 0.60.