Math 026L.04 Spring 2002
Test #2 Solutions

1.

(a) $y(x) = \ln(2 + x) - \ln 2$

(b) $y(x) = \tan(\tan x)$

2.

(a) $S(t) = 150\, e^{-\frac{1}{50}t}$ so that $S(20) = 100.55$ kg.

(b) After a very long time the concentration is 0 kg/L. This is not a surprise because we would expect the pure water that is entering to eventually flush all of the sugar out of the tank.

(c) $S(t) = -350\, e^{-\frac{1}{50}t} + 500$ so that $S(20) = 265.39$ kg.

(d) After a very long time the concentration is 0.5 kg/L. This is not a surprise because we would expect the sugar water that is entering to eventually flush all of the original sugar out of the tank. Thus, all that would remain is the sugar that enters at a rate of 0.5 kg/L.

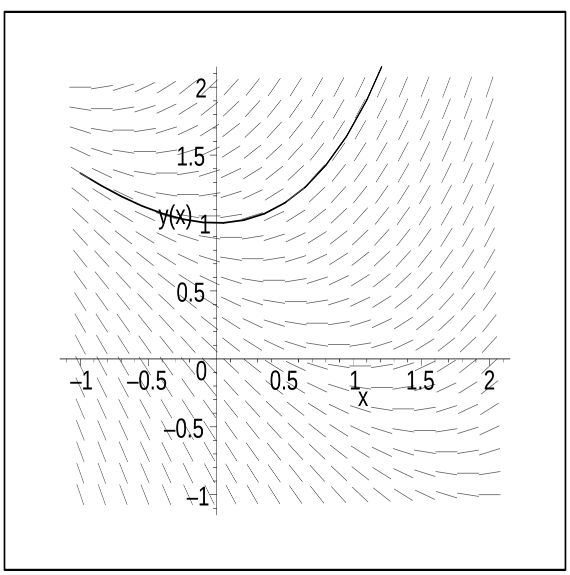
3.

(a) $\dfrac{dP}{dt} = .04P$ so $P(t) = 1400e^{.04t}$ and $P(12) = 2263$. There cannot be fractional people.

(b) $\dfrac{dP}{dt} = .04P + 30$ so $P(t) = 2150e^{.04t} - 750$ and $P(12) = 2725$.

4.

(a)



(c) $y(0) = 1$, $y(1/2) = 1$, $y(1) = 5/4$, $y(3/2) = 15/8$, $y(2) = 49/16$.

5.

(a) $P = 0$ and $P = 2000$ are stable equilibria while $P = 1000$ is an unstable equilibrium.

(b) If $P(0) = 900$, then the population goes extinct because for any initial value $P(0) < 1000$, the solution curve approaches 0.

(c) If $P(0) = 1100$, then the population does not go extinct because for any initial value $1000 < P(0) < 2000$, the solution curve approaches 2000.

6.

(a) $C(t) = C_0 e^{-\beta t}$

(b) Substituting our solution from **(a)** yields $\dfrac{dS}{dt} = \alpha SC = \alpha SC_0 e^{-\beta t}$. Separating variables gives $\dfrac{1}{S}\dfrac{dS}{dt} = \alpha C_0 e^{-\beta t}$. Anti-differentiating we have $\ln S = \dfrac{\alpha C_0}{-\beta} e^{-\beta t} + K$ for some arbitrary constant K.

If $S(0) = S_0$ then we get $\ln S_0 = -\dfrac{\alpha C_0}{\beta} + K$ or $K = \ln S_0 + \dfrac{\alpha C_0}{\beta}$.

Thus,

$$\ln S = -\frac{\alpha C_0}{\beta} e^{-\beta t} + K = -\frac{\alpha C_0}{\beta} e^{-\beta t} + \ln S_0 + \frac{\alpha C_0}{\beta} = \frac{\alpha C_0}{\beta}(1 - e^{-\beta t}) + \ln S_0.$$

Hence,

$$S(t) = S_0 \, e^{\frac{\alpha C_0}{\beta}(1-e^{-\beta t})}.$$

(c) $\lim_{t\to\infty} S(t) = S_0 \, e^{\frac{\alpha C_0}{\beta}}$

7.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k} = 0.693$ accurate to three decimal places. Notice that the terms in the sum are decreasing. Since all the terms for $k \geq 11$ are smaller than 0.00009 as we add more terms, we will not affect the first 3 decimal places.

(c) $\ln 2 = .693$ accurate to three decimal places. This tells us that we can write $\ln 2$ as the infinite sum

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k2^k}.$$

(Note: The Lunar New Year (lny) posters that were pinned around campus a few weeks ago attempted to write a more general version of this formula. However, the author of that poster made a mistake!)