Math 026L.04 Spring 2002
Modeling with Differential Equations

Logistic Equation: The differential equation $\frac{dy}{dt} = cy(M-y)$ where c and M are constants is called the logistic equation. The logistic equation can be solved by separation of variables, but it is not apparent.

In fact, if $y(0) = y_0$, then the solution is

$$y = \frac{y_0 M}{y_0 + (M - y_0)e^{-cMt}}.$$

The **carrying capacity** is the maximum population that the environment is capable of sustaining in the long run.

Suppose $P(t)$ represents a population at time t. The **relative growth rate** is given by $\frac{1}{P}\frac{dP}{dt}$.

If the relative growth rate is constant, then the population is exponential.

$$\frac{1}{P}\frac{dP}{dt} = k \Longrightarrow \frac{dP}{dt} = kP \Longrightarrow P(t) = P_0 e^{kt}$$

If the relative growth rate is (decreasing) linear, then the population is logistic.

$$\frac{1}{P}\frac{dP}{dt} = k - aP \Longrightarrow \frac{dP}{dt} = P(k - aP) \Longrightarrow \frac{dP}{dt} = aP\left(\frac{k}{a} - P\right) \Longrightarrow P(t) = \frac{\frac{k}{a}Ce^{kt}}{1 + Ce^{kt}}$$

1. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity?

(b) Sketch a slope field for this differential equation.

(c) What are the equilibrium solutions? Identity each as being either stable or unstable?

(d) Sketch solution curves corresponding to initial populations of 20, 70, 100, 130. Which solutions have inflection points? At what population levels do they occur?

2. The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	0	2	4	6	8	10	12	14	16	18
Yeast cells	18	39	80	171	336	509	597	640	664	672

(a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.

(b) Use the data to estimate the initial relative growth rate.

(c) Find both an exponential model and a logistic model for these data.

(d) Compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.

(e) Use your logistic model to estimate the number of yeast cells after 7 hours.