Mathematics 103.04 and 103.05 **Test Four**

I have neither given nor received aid in the completion of this test.
Signature:

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To get full credit you must show enough work to convince me that you know what you are doing!

1. 10 pts. Let $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$. Compute the divergence and curl of $\mathbf{F}$.

Solution.

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}xy + \frac{\partial}{\partial z}xz = 4x.$$

$$\nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & xz \end{bmatrix} = -z\mathbf{j} + y\mathbf{k}.$$

2. 10 pts. Let $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$. Exhibit a function f such that $\mathbf{F} = \nabla f$.

Solution. Fix $(a, b) \in \mathbf{R}^2$. Let C be the curve $x = at,\ y = bt,\ 0 \le t \le 1$. The function f with gradient $\mathbf{F}$ whose value at $(0, 0)$ is zero is then given by

$$f(a, b) = \int_C y\,dx + (x + 2y)\,dy = \int_0^1 (bt)\,d(at) + (at + 2bt)\,d(bt) = ab + b^2.$$

3. 20 pts. Let $\mathbf{F} = y\mathbf{i}$ and let C be the boundary of the square with vertices $(0, 0), (0, 1), (1, 1)$ and $(1, 0)$ traversed counterclockwise. Evaluate

$$\int_C \mathbf{F} \bullet \mathbf{T}\,ds$$

from the definition of a line integral and then use Green's Theorem to evaluate it.

Solution. Using the definition, we first write

$$\int_C \mathbf{F} \bullet \mathbf{T}\,ds = \int_C y\,dx.$$

On the two vertical edges dx is zero and on the bottom edge y is zero. The top edge may be oppositely parameterized by $x = t,\ y = 1,\ 0 \le t \le 1$ so

$$\int_C y\,dx = -\int_0^1 1\,dt = -1.$$

Green's Theorem says the line integral is the integral of the curl of $\mathbf{F}$ over the square; this curl is -1 so the integral over the square is also -1 as the square has area 1.

4. 15 pts. Use Green's Theorem to express

$$\int_C xy\,dx\,+\,x^2\,dy$$

as an iterated integral where C is the first quadrant loop of the curve with polar eqation $r = \sin\,2\theta$ traversed in the counterclockwise sense.

Solution. The curl of $xy\mathbf{i} + x^2\mathbf{j}$ is x so Green's Theorem says that line integral in question is

$$\iint_R x\,dxdy$$

where R is the interior of the loop. Changing to polar coordinates, we may express this double integral as

$$\int_0^{\pi/2}\int_0^{\sin\,2\theta}(r\cos\theta)\,r\,dr\,d\theta.$$

5. 15 pts. Express

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS$$

as an iterated integral; here $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 3\mathbf{k}$, S is the part of $z = 4 - x^2 - y^2$ lying above the xy-plane and $\mathbf{n}$ is the *downward* pointing unit normal to S.

Solution. We may parameterize S by

$$\mathbf{r}(x,y)\;=\;(x,y,4-x^2-y^2),\;\;(x,y)\in R\;=\;\{(x,y):x^2+y^2\le 4\}.$$

We then have

$$\mathbf{r}_x\times\mathbf{r}_y\;=\;(2x,2y,1)\text{ for }(x,y)\in R.$$

Thus, as $\mathbf{r}_x\times\mathbf{r}_y$ points *up*, the surface integral is given by

$$-\iint_R(2x,2y,3)\bullet(2x,2y,1)\,dxdy\;=\;-\iint_R 4x^2+4y^2+3\,dxdy\;=\;-\int_0^{2\pi}\int_0^2(4r^2+3)\,r\,dr\,d\theta.$$

6. 15 pts. Use the Divergence Theorem to express

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS$$

as an iterated integral; here $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, S is the boundary of the solid bounded by the coordinate planes and the the plane $x + y + z = 1$ and where $\mathbf{n}$ is unit normal to S which points out of this solid.

Solution. The divergence of $\mathbf{F}$ is zero so the the answer is zero. A more interesting problem, and the one I intended to give, is when $\mathbf{F} = xyz\mathbf{k}$, which I now solve. Let T be the solid. We have $\nabla\bullet\mathbf{F} = xy$ so, by the Divergence Theorem, the surface integral equals

$$\iiint_T xy\,dxdydz\;=\;\int_0^1\int_0^{1-x}\int_0^{1-x-y}xy\,dx\,dy\,dz.$$

7. 30 pts. Evaluate both sides of Stokes' Theorem where S is part of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = 2$ with unit normal $\mathbf{n}$ pointing toward the z-axis and where $\mathbf{F} = z^2\mathbf{j}$. (The surface integral neet only be expressed as an iterated integral and the line integral need only be expressed as a definite integral.)

Solution. Let $C^{\mathbf{top}}$ be the circle where $z = 2$ oriented in the counterclockwise sense and let $C^{\mathbf{bot}}$ be the circle where $z = 0$ oriented in the clockwise sense. Note that because $\mathbf{n}$ points inward that these are the correct orientations for Stokes' Theorem. Because $\mathbf{F}$ vanishes on $C^{\mathbf{bot}}$,

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds \;=\; \int_{C^{\mathbf{top}}} \mathbf{F} \bullet \mathbf{T}\, ds \;=\; \int_{C^{\mathbf{top}}} z^2\, dy \;=\; 2^2 \int_0^{2\pi} d(2\sin\theta) \;=\; 0.$$

On the other hand, we may parameterize S by

$$\mathbf{r}(\theta, z) \;=\; (\cos\theta, \sin\theta, z),\;\; (\theta, z) \in R = (0, 2\pi) \times (0, 2);$$

in class we found that

$$\mathbf{r}_\theta \times \mathbf{r}_z(\theta, z) \;=\; (\cos\theta, \sin\theta, 0)$$

for $\theta, z) \in R$. Thus, as this cross product points away from the z-axis, we find that

$$\iint_S \nabla \times \mathbf{F} \bullet \mathbf{n}\, dS \;=\; -\iint_R (0, -2z, 0) \bullet (\cos\theta, \sin\theta, 0)\, dz\, d\theta \;=\; -\int_0^{2\pi} \int_0^2 -2z \sin\theta\, dz\, d\theta \;=\; 0.$$

That's all folks!