Mathematics 103.04 and 103.05 **Test Two**

I have neither given nor received aid in the completion of this test.
Signature:

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To get full credit you must show enough work to convince me that you know what you are doing!

1. 10 pts. Does

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$

exist? As it says above, to get *full* credit you must justify your answer; you get half credit if your answer is correct but your justification is not.

Solution. Let $r = \sqrt{x^2+y^2+z^2}$. We have

$$|\frac{xyz}{x^2+y^2+z^2}| \;=\; |\frac{x}{r}\frac{y}{r}\frac{z}{r}|r \;\le r.$$

Let $\epsilon > 0$. It follows that

$$r = |(x,y,z)-(0,0,0)| < \epsilon \;\Rightarrow |\frac{xyz}{x^2+y^2+z^2} - 0| < \epsilon$$

so the limit exists and is zero.

2. 5 pts. Let $f(x,y) = x^2y$. Find

$$\lim_{h\to 0} \frac{f(1+h,2+3h)-f(1,2)}{h}.$$

Solution. Let $x(t) = 1+t$ and let $y(t) = 2+3t$. The above limit is just

$$\frac{d}{dt} f(x(t),y(t)\Big| t=0$$

which by the chain rule is

$$\frac{\partial f}{\partial x}(1,2)x'(0) + \frac{\partial f}{\partial y}(1,2)y'(0) \;=\; 4\,1 + 1\,3 \;=\; 7$$

as $\frac{\partial f}{\partial x}(x,y) = 2xy$ so $\frac{\partial f}{\partial x}(1,2) = 4$ and $\frac{\partial f}{\partial y}(x,y) = x^2$ so $\frac{\partial f}{\partial x}(1,2) = 1$.

3. 5 pts. Find an equation for the tangent plane to the graph of $f(x,y) = xy$ at $(1,2,2)$.

Solution. Let $F(x,y,z) = z - xy$. Note that $\nabla F(x,y,z) = (-y,-x,1)$. The normal to the tangent plane is $\nabla F(1,2,2) = (-2,-1,1)$ so an equation for the plane is

$$-2(x-1) - (y-2) + (z-1) \;=\; 0.$$

4. 5 pts. Find an equation for the the tangent plane to $\{(x,y,z) : xyz = 6\}$ at $(1,2,3)$.

Solution. Let $F(x,y,z) = xyz$. Note that $\nabla F(x,y,z) = (yz, xz, xy)$. The normal to the tangent plane is $\nabla F(1,2,3) = (6,3,2)$ so an equation for the plane is

$$6(x-1) + 3(y-2) + 2(z-3) \;=\; 0.$$

5. 10 pts. Suppose F is a function of two variables such that

$$\frac{\partial F}{\partial x}(x,y) \;=\; y^2 + 1, \qquad \frac{\partial F}{\partial y}(x,y) \;=\; 2xy.$$

Calculate

$$\frac{d}{dt} F(\cos t, \sin t)\Big|_{t=\pi/2}.$$

Solution. By the chain rule, the answer is

$$\frac{\partial F}{\partial x}(\cos t, \sin t)\Big|_{t=\pi/2} \frac{d}{dt}\cos t\Big|_{t=\pi/2} \;+\; \frac{\partial F}{\partial y}(\cos t, \sin t)\Big|_{t=\pi/2} \frac{d}{dt}\sin t\Big|_{t=\pi/2} \;=\; -2.$$

6. 10 pts. Suppose F is a function of three variables such that

$$\frac{\partial F}{\partial x}(3,5,2) \;=\; 1, \qquad \frac{\partial F}{\partial y}(3,5,2) \;=\; 2, \qquad \frac{\partial F}{\partial z}(3,5,2) \;=\; 3.$$

Suppose

$$x(u,v) \;=\; u+v, \qquad y(u,v) \;=\; u^2+v^2, \qquad z(u,v) \;=\; uv.$$

Let $f(u,v) = F(x(u,v), y(u,v), z(u,v))$. Calculate

$$\frac{\partial f}{\partial u}(1,2), \qquad \frac{\partial f}{\partial v}(1,2).$$

Solution. Note that $(x(1,2), y(1,2), z(1,2)) = (3,5,2)$. By the chain rule,

$$\frac{\partial f}{\partial u}(1,2) \;=\; \frac{\partial F}{\partial x}(3,5,2)\frac{\partial x}{\partial u}(1,2) \;+\; \frac{\partial F}{\partial y}(3,5,2)\frac{\partial y}{\partial u}(1,2) \;+\; \frac{\partial F}{\partial z}(3,5,2)\frac{\partial z}{\partial u}(1,2) \;=\; 1\cdot 1 + 2\cdot 2 + 3\cdot 2 \;=\; 10$$

and

$$\frac{\partial f}{\partial v}(1,2) \;=\; \frac{\partial F}{\partial x}(3,5,2)\frac{\partial x}{\partial v}(1,2) \;+\; \frac{\partial F}{\partial y}(3,5,2)\frac{\partial y}{\partial v}(1,2) \;+\; \frac{\partial F}{\partial z}(3,5,2)\frac{\partial z}{\partial v}(1,2) \;=\; 1\cdot 1 + 2\cdot 4 + 3\cdot 1 \;=\; 12$$

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7. 10 pts. Determine the point(s) on $\{(x,y) : x^2 + y^2 = 1\}$ where $f(x,y) = x^2 y$ attains its maximum and minimum values.

Solution. Let $F(x,y) = x^2 + y^2$. By the method of Lagrange multipliers,

$$\nabla f(x,y) \;=\; \lambda \nabla F(x,y)$$

for some λ at any maximum point (x, y) which amounts to

$$2xy = 2\lambda x \qquad \text{and} \qquad x^2 = 2\lambda y.$$

If we allow $x = 0$ then $y = \pm 1$ and $f(x, y) = 0$. If $x \neq 0$ then the first equation gives $\lambda = y$ which, when substituted in the second equation, gives $x^2 = 2y^2$ or $y = \pm x/\sqrt{2}$. Keeping in mind that $x^2 + y^2 = 1$ we obtain the points $(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}), (-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}), (-\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}), (-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}})$ as candidates for maximum/minimum points on which f is $\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}, -\frac{2}{3\sqrt{3}}, -\frac{2}{3\sqrt{3}}$, respectively. Thus the maxima are the first two and the minima are the last two of these four points.

Alternatively, you can find the points where $\nabla(x^2 y) \times \nabla(x^2 + y^2) = \mathbf{0}$ and $x^2 + y^2 = 1$.

8. 20 pts. Let $T = \{(x, y) : y \geq 0,\ y \leq -2x + 2 \text{ and } y \leq 2x + 2\}$; note that T is a closed triangle with vertices $(-1, 0)$, $(0, 2)$, $(1, 0)$. Let $f(x, y) = xy - x$. Determine the point(s) on T where f attains its maximum and minimum values.

Solution. We have that $\partial f/\partial x(x, y) = y - 1$ and $\frac{\partial f}{y}(x, y) = x$ so $(0, 1)$ is the unique critical point of f. It is a candidate for a maximum/minimum as it is interior to T.

$(0, 1) \ni x \mapsto (x, -2x + 2)$ parameterizes the upper right hand side of T and

$$\frac{d}{dx} f(x, -2x + 2) = \frac{d}{dx} x(-2x + 2) - x = -4x + 1$$

which is zero when $x = 1/4$ so $(1/4, 3/2)$ is a candidate for a maximum/minimum.

$(-1, 0) \ni x \mapsto (x, 2x + 2)$ parameterizes the upper left hand side of T and

$$\frac{d}{dx} f(x, 2x + 2) = \frac{d}{dx} x(2x + 2) - x = 4x + 1$$

which is zero when $x = -1/4$ so $(-1/4, 3/2)$ is a candidate for a maximum/minimum.

$f(x, 0) = -x$ on the x-axis so there are no maxima/minima ont the bootom of T. The three vertices $(-1, 0), (0, 2), (1, 0)$ are candidates for maxima/minima.

Now $f(0, 1) = 0$, $f(1/4, 3/2) = 1/8$, $f(-1/4, 3/2) = -1/8$, $f(-1, 0) = 1$, $f(0, 2) = 0$ and $f(1, 0) = -1$. Thus $(1, 0)$ is the unique minimum and $(0, 2)$ is the unique maximum for f on T

9. 15 pts. Find the point(s) at which z is largest on $\{(x, y, z) : x^2 + 2y^2 + 3z^2 = 1 \text{ and } x + 2y + 3z = 0\}$.

Solution. Let $f(x, y, z) = z$, $F(x, y, z) = x^2 + y^2 + z^2$ and let $G(x, y, z) = x + 2y + 3z$. By the method of Lagrange multipliers, if (x, y, z) is a maximum/minimum on the set where $F = 1$ and $G = 0$ then

$$\nabla f(x, y, z) = \lambda \nabla F(x, y, z) + \mu \nabla G(x, y, z)$$

for some λ, μ, which amounts to

$$0 = 2\lambda x + \mu, \qquad 0 = 4\lambda y + 2\mu, \qquad 1 = 6\lambda z + 3\mu.$$

Note that it is impossible that $\lambda = 0$. The first equation says that $\mu = -2\lambda x$ which, when substituted in the second equation gives $0 = 4\lambda y - 4\lambda x$ so, as $\lambda \neq 0$, gives $x = y$. Since $x + 2y + 3z = 0$ we find that $z = -x$. Since $x^2 + 2y^2 + 3z^2 = 1$ we find that $x = \pm 1/\sqrt{6}$. Thus the points $1/\sqrt{6}(1, 1, -1)$ and $1/\sqrt{6}(-1, -1, 1)$ are the only possibilities for maxima/minima. It follows that $1/\sqrt{6}(-1, -1, 1)$ is the unique maximum point for z on $F + 1$ and $G = 0$.

Alternatively, you can find the point(s) where $F = 1$, $G = 0$ and where the triple product $[\nabla f, \nabla F, \nabla G]$ is zero.

That's all folks!