Mathematics 103.04 and 103.05 **Test One**

I have neither given nor received aid in the completion of this test.
Signature:

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To get full credit you must show enough work to convince me that you know what you are doing!

1. 25 pts. Let $\mathbf{a} = (3, 0, 4)$ let $\mathbf{b} = (-1, -1, -1)$ and let $\mathbf{c} = (4, 0, -1)$. Compute:

$$\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}, \quad \mathbf{a} \bullet \mathbf{b}, \quad \mathbf{a} \times \mathbf{c}, \quad |3\mathbf{c}|, \quad [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

Solution.

$$\begin{aligned}
\mathbf{a} + 3\mathbf{b} - 4\mathbf{c} &= (3, 0, 4) + (-3, -3, -3) + (-16, 0, 4) = (-16, -3, 5); \\
\mathbf{a} \bullet \mathbf{b} &= -7 \\
\mathbf{a} \times \mathbf{c} &= (0, 19, 0) \\
|3\mathbf{c}| &= 3\sqrt{17} \\
[\mathbf{a}, \mathbf{b}, \mathbf{c}] &= 19.
\end{aligned}$$

2. 10 pts. Write down an equation whose solution set is the plane containing the three points $(1, 2, 3), (3, 2, 1), (1, 1, 0)$.

Solution. A normal to this plane is

$$((1, 2, 3) - (1, 1, 0)) \times ((3, 2, 1) - (1, 1, 0)) = (0, 1, 3) \times (2, 1, 1) = (-2, 6, -2)$$

and a vector in the plane is $(1, 1, 0)$ so if $\mathbf{x} = (x, y, z)$ is any point in the plane we have

$$(\mathbf{x} - (1, 1, 0)) \bullet (-2, 6, -2) = 0$$

or

$$-2x + 6y - 2z = 4.$$

3. 10 pts. Let L be the line $\{(1, 1, 0) + t(1, 2, 3) : t \in \mathbf{R}\}$ and let $\mathbf{c} = (3, 2, 1)$. Show that $\mathbf{c} \notin L$ and write down an equation whose solution set is the plane containing L and $\mathbf{c}$.

Solution. Suppose $\mathbf{c} = (1, 1, 0) + t(1, 2, 3)$ for some $t \in \mathbf{R}$. Then $3 = 1 + t,\ 2 = 1 + 2t, 1 = 3t$ which is clearly impossible so $\mathbf{c} \in L$.

The sought after plane contains the three points $(1, 1, 0)$ $(t = 0)$, $(2, 3, 3)$ $(t = 1)$ and $(3, 2, 1)$. Thus a normal to this plane is

$$((2, 3, 3) - (1, 1, 0)) \times ((3, 2, 1) - (1, 1, 0)) = (1, 2, 3) \times (2, 1, 1) = (-1, 5, -3).$$

Thus the plane is the solution set of

$$(\mathbf{x} - (1,1,0)) \bullet (-1,5,-3) \;=\; 0$$

where $\mathbf{x} = (x,y,z)$ or

$$-x + 5y - 3z \;=\; 4.$$

4. 15 pts. Let

$$L \;=\; \{\mathbf{x} \in \mathbf{R}^3 : \mathbf{x} \times (1,1,0) = (2,-2,1)\}.$$

I tell you that L is a line and ask you to exhibit vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \{\mathbf{a} + t\mathbf{b} : t \in \mathbf{R}\}$.

Solution. Set $\mathbf{x} = (x,y,z)$ and calculate $\mathbf{x} \times (1,1,0) = (-z, z, x-y)$; this vector equals $(2,-2,1)$ if and only if $z = -2$ and $x - y = 1$. Thus $\mathbf{x} \in L$ if and only if $(x,y,z) = (x, x-1, -2) = (0,-1,-2) + (1,1,0)$. So we can take $\mathbf{a} = (0,-1,-2)$ and $\mathbf{b} = (1,1,0)$.

5. 10 pts. Let the plane curve $\mathbf{P}$ be defined by setting $\mathbf{P}(t) = (2e^t, e^{2t})$ for $t \in \mathbf{R}$. Determine a real valued function f of a real variable whose graph contains the range of $\mathbf{P}$. (In the parlance of the book, $x = 2e^t$ and $y = e^{2t}$ and you have to eliminate t.)

Solution. Set $x = 2e^t$ and $y = e^{2t}$. Then $y = e^{2t} = (e^t)^2 = (x/2)^2$ so we can set $f(x) = (x/2)^2$ for $x \in \mathbf{R}$.

6. 15 pts. Prove or disprove:

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b}) \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbf{R}^3.$$

Solution. Using properties of the cross product we obtain

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) \;=\; \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} \;=\; \mathbf{0} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} - \mathbf{0} \;=\; 2(\mathbf{a} \times \mathbf{b})$$

7. 25 pts. Let $\mathbf{P}(t) = (t, t^2, t^3)$ for $0 < t < \infty$. Compute the velocity, speed, acceleration and curvature of $\mathbf{P}$. Say what how the curvature behaves as $t \to \infty$.

Solution. We have

$$\begin{aligned}
\frac{d\mathbf{P}}{dt} &= (1, 2t, 3t^2); \\
\left|\frac{d\mathbf{P}}{dt}\right| &= \sqrt{1 + 4t^2 + 9t^4}; \\
\frac{d^2\mathbf{P}}{dt^2} &= (0, 2, 6t); \\
\kappa &= \frac{\left|\frac{d\mathbf{P}}{dt} \times \frac{d^2\mathbf{P}}{dt^2}\right|}{\left|\frac{d\mathbf{P}}{dt}\right|^3} \;=\; \frac{|(6t^2, -6t, 2)|}{(1 + 4t^2 + 9t^4)^{3/2}} \;=\; \frac{(36t^4 + 36t^2 + 4)^{1/2}}{(1 + 4t^2 + 9t^4)^{3/2}}.
\end{aligned}$$

In the curvature, as $t \to \infty$, the dominant term in the numerator is t and the dominant term in the denominator is t^2; thus the curvature tends to zero as $t \to \infty$.

8. 25 pts. Suppose $\mathbf{a}$ and $\mathbf{b}$ are distinct vectors in $\mathbf{R}^3$ and r and s are positive numbers such each of which is less than $|\mathbf{b} - \mathbf{a}|$ and whose sum is greater than $|\mathbf{b} - \mathbf{a}|$. Let $R = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x} - \mathbf{a}| = r\}$ and let

$S = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x} - \mathbf{b}| = s\}$. I tell you that $R \cap S$ is a circle. Determine the center and radius of this circle as well as a vector which is normal to the plane containing this circle.

Solution. Let $\mathbf{c}$ be the center of the circle. Then $\mathbf{c}$ lies on the segment joining $\mathbf{a}$ to $\mathbf{b}$ so $\mathbf{c} = (1-t)\mathbf{a} + t\mathbf{b}$ for some $t \in (0,1)$. If $\mathbf{x}$ is a point on the circle we have from the Pythagorean Theorem that

$$r^2 = |\mathbf{x} - \mathbf{a}|^2 = |\mathbf{x} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 \qquad \text{and} \qquad s^2 = |\mathbf{x} - \mathbf{b}|^2 = |\mathbf{x} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{b}|^2.$$

Subtracting these two and noting that $|\mathbf{c} - \mathbf{a}| = t|\mathbf{b} - \mathbf{a}|$ and $|\mathbf{c} - \mathbf{b}| = (1-t)|\mathbf{b} - \mathbf{a}|$ we find that

$$r^2 - s^2 \;=\; (t^2 - (1-t)^2)|\mathbf{b} - \mathbf{a}|^2 \;=\; (2t-1)|\mathbf{b} - \mathbf{a}|^2$$

so

$$t \;=\; \frac{1}{2}\big(r^2 - s^2 + |\mathbf{b} - \mathbf{a}|^2\big)$$

The square of the radius of the circle is

$$r^2 - |\mathbf{c} - \mathbf{a}|^2 \;=\; s^2 - |\mathbf{c} - \mathbf{b}|^2$$

and a normal to the plane of the circle is $\mathbf{b} - \mathbf{a}$.

That's all folks!