Math 111.01 Summer 2003
July 16, 2003

Related rates problems are those in which, appropriately enough, the rates of change of two different quantities are related. Usually one rate of change is given in the problem and the other rate of change is asked for. The key to solving such problems lies in finding an equation relating the two quantities whose rates of change are involved, and then differentiating this equation to obtain a relationship between the rates of change. Some problems may involve implicit differentiation (depending on the equation involved). Be careful that you use the chain rule properly when you differentiate as in the example below.

Partial Example: Suppose you are given a problem involving the area A and side length s of a square region whose size is changing. If A changes with respect to time, s will change with it as well (i.e., if the area is getting larger, then the sides must also be getting larger), and vice-versa. Thus, both the area and the side length of the square are functions of time, $A = A(t)$ and $s = s(t)$. An equation relating the area and side length of a square is $A = s^2$. We can differentiate this equation with respect to time t to get

$$\frac{dA}{dt} = 2s\,\frac{ds}{dt}.$$

Note that we used the chain rule when we differentiated s^2 since s is a function of t, and thus $(s(t))^2$ is the composition of two functions. Now, given a particular instant in time, say the instant when the side length of the square is 3 inches, we can use information about the rate of change of the area at that moment to determine the rate of change of the side length at that moment, and vice-versa.

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1. Supose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates $\frac{dS}{dt}$ to $\frac{dr}{dt}$.

2. The radius r and height h of a right circular cone are related to the cone's volume V by the formula $V = (1/3)\pi r^2 h$.

(a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if r is constant?

(b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if h is constant?

(c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither r nor h is constant?

3. The length l of a rectangle is decreasing at the rate of 2 cm/s while the width w is increasing at the rate of 2 cm/s.

(a) When $l = 12$ cm and $w = 5$ cm, what is the rate of change of the area of the rectangle? Is it increasing or decreasing?

(b) When $l = 12$ cm and $w = 5$ cm, what is the rate of change of the perimeter of the rectangle? Is it increasing or decreasing?

4. A Japanese beetle infestation is spreading from the center of a small town. Since beetles fly in all directions, we assume the region they cover is circular. Suppose the radius of this circular region is increasing at a rate of 3 miles per year. Determine the rate of change of the area of infestation when the radius is 4 miles.

5. A manufacturer of tennis balls decides to increase production by 30 cans each day. The manufacturer has determined that the total revenue R (in dollars) from the sale of x cans of tennis balls in a day is approximately given by the function $R(x) = 2.14x - 0.0003x^2$. Determine the rate of change of revenue with respect to time when the daily production level is 1200 cans. (Assume all cans are sold.)

6. Suppose that a gas balloon is being filled at the rate of 300 cubic centimeters of helium per second. At what rate is the radius of the balloon increasing when the radius is 100 cm? Compare your two answers and explain why the *relative* sizes are plausible.

7. Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the radius of the drop increases at a constant rate.

8. Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates: $\dfrac{dx}{dt} = 1$ m/s, $\dfrac{dy}{dt} = -2$ m/s, $\dfrac{dz}{dt} = 1$ m/s.

(a) Find the rate at which the volume of the box is changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

(b) Find the rate at which the surface area of the box is changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

9. A 13 foot ladder is leaning against a house when its base starts to slide away. By the time the base is 12 feet from the house, the base is moving (horizontally away from the wall) at the rate of 5 ft/s.

(a) How fast is the top of the ladder sliding down the wall then?

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

10. Two commercial airplanes are flying at 40,000 feet along straight-line courses that intersect at right angles. Plane A is approaching the intersections point at a speed of 442 knots (nautical miles per hour). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

11. Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile (i.e., a pile that looks like an upside-down ice cream cone). The height of the pile is always three-eighths of the base diameter.

(a) How fast is the height of the pile changing when the pile is 4 meters high? Your answer is probably in m/min; convert your answer to cm/min.

(b) How fast is the radius of the pile changing when the pile is 4 meters high? Your answer is probably in m/min; convert your answer to cm/min.

12. Water is draining at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down, i.e., the ice cream cone is right-side up). The conical reservoir has a top radius of 45 meters and a height of 6 meters.

(a) How fast is the water level falling when the water is 5 meters deep? (Convert your answer to cm/min.)

(b) How fast is the radius of the water's surface changing then? (Again, convert your answer to cm/min.)

13. When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

14. A spherical balloon is inflated with helium at the rate of 100π cubic feet per minute.

(a) How fast is the balloon's radius increasing at the instant the radius is 5 feet?

(b) How fast is the surface area increasing at the same instant?

15. A man who is 6 feet tall walks at the rate of 5 ft/s toward a streetlight that is 16 feet above the ground.

(a) At what rate is the tip of his shadow moving?

(b) At what rate is the length of his shadow changing when he is 10 feet from the base of the streetlight?

16. A spherical iron ball 8 inches in diameter is coated with a layer of ice of uniform thickness.

(a) If the ice melts at the rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 inches thick?

(b) How fast is the outer surface area of ice decreasing at that time?

Answers to selected problems

3. **(a)** 14 cm^2/s, increasing; **(b)** 0 cm/s, neither (i.e., constant).

9. **(a)** -12 ft/s; **(b)** -59.5 ft^2/s.

11. **(a)** 11.19 cm/min; **(b)** 14.92 cm/min.

14. **(a)** 1 ft/min; **(b)** 40π ft^2/min.

16. **(a)** $5/(72\pi)$ in/min; **(b)** 10/3 in^2/min.