Math 111.01 July 17, 2003
Summer 2003

Derivatives and Shapes of Curves (4.3)

Recall from Section 2.10 that the derivative tells us information about the shape of a curve.

First Derivative

Fact. If $f'(c) = 0$, then f has a horizontal tangent at c.

Definition. f has a *critical number* at c in $\mathscr{D}(f)$ if either $f'(c) = 0$ or $f'(c)$ DNE.

Fact (Fermat's Theorem). If f has a local extremum at c, then c is a critical point of f.

Increasing/Decreasing Test

(a) If $f' > 0$ on an interval, then f is increasing on that interval.
(b) If $f' < 0$ on an interval, then f is dereasing on that interval.

First Derivative Test

Suppose that c is a critical number of the continuous function f.

(a) If f' changes sign from positive to negative at c, then f has a local maximum at c.
(b) If f' changes sign from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c, then f has no local extremum at c.

Hint: Remember all of these with a picture.

Second Derivative

Definition. A function f is *concave up* on an interval I if f' is increasing on I.

Definition. A function f is *concave down* on an interval I if f' is decreasing on I.

Definition. A point c where f changes concavity is called an *inflection point*.

Concavity Test

(a) If $f'' > 0$ on an interval, then f is concave up on that interval.
(b) If $f'' < 0$ on an interval, then f is concave down on that interval.

Second Derivative Test

Suppose that f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example. $f(x) = |x|$ has a local minimum at 0, but $f'(0)$ DNE.

Example. $f(x) = x^{1/3}$ changes concavity at 0 so that 0 is an inflection point, but $f'(0)$ and $f''(0)$ DNE.

Example. Sketch a graph of $f(x) = \dfrac{x^2 + 7x + 12}{x^2}$ being as detailed as possible.

Solution: We collect information from f, f', and f''.

• ***f* tells us roots and asymptotes**

(i) The domain of f is $\mathscr{D}(f) = \{x \neq 0\}$.

(ii) Roots occur where $f(x) = 0$. Thus, $f(x) = 0$ when $x^2 + 7x + 12 = 0$ (provided that $x \neq 0$). Since, $x^2 + 7x + 12 = (x + 4)(x - 3)$, there are two roots, namely $x = -4$ and $x = 3$.

(iii) To compute horizontal asymptotes, we need to check $\lim_{x\to\pm\infty} f(x)$. Thus,

$$\lim_{x\to\infty} \frac{x^2 + 7x + 12}{x^2} = \lim_{x\to\infty} (1 + 7x^{-1} + 12x^{-2}) = 1$$

and

$$\lim_{x\to-\infty} \frac{x^2 + 7x + 12}{x^2} = \lim_{x\to-\infty} (1 + 7x^{-1} + 12x^{-2}) = 1$$

so that $y = 1$ is a horizontal asymptote.

(iv) To compute vertical asymptotes, we check the behaviour where f blows up.

$$\lim_{x\to 0+} \frac{x^2 + 7x + 12}{x^2} = \infty$$

and

$$\lim_{x\to 0-} \frac{x^2 + 7x + 12}{x^2} = \infty.$$

(v) We can also find the "sign" of f:

$$\begin{array}{rl} x < -4: & f > 0 \\ -4 < x < -3: & f < 0 \\ -3 < x < 0: & f > 0 \\ x > 0: & f > 0 \end{array}$$

• f' **tells us CNs and intervals of increase/decrease**

We compute

$$f'(x) = \frac{d}{dx}\frac{x^2+7x+12}{x^2} = \frac{d}{dx}(1+7x^{-1}+12x^{-2}) = -7x^{-2}-24x^{-3} = \frac{-(7x+24)}{x^3}.$$

Do not forget to write f' in factored form.

CNs occur where $f'(x)=0$ or where f' DNE. Thus, $f'(x)=0$ when $x=-24/7$, and f' DNE when $x=0$. Observe that $0 \notin \mathscr{D}(f)$ so it is not a CN.

(vi) The intervals of increase and decrease are:

$$\begin{array}{ccl} x < -24/7: & f'<0: & f \text{ decreasing} \\ -24/7 < x < 0: & f'>0: & f \text{ increasing} \\ x > 0: & f'<0: & f \text{ decreasing} \end{array}$$

• f'' **tells us concavity**

We compute

$$f''(x) = \frac{d}{dx}\frac{-(7x+24)}{x^3} = \frac{d}{dx}(-7x^{-2}-24x^{-3}) = 14x^{-3}+72x^{-4} = \frac{14x+72}{x^4}.$$

Do not forget to write f'' in factored form.

Notice that $f''(x)=0$ when $x=-72/14=-36/7$ and that f'' DNE when $x=0$.

(vii) The concavity is:

$$\begin{array}{ccl} x < -36/7: & f''>0: & f \text{ is concave down} \\ x > -36/7: & f''<0: & f \text{ is concave up} \end{array}$$

Hence, the only inflection point is $x=-36/7$.

• **combine and sketch** f

Mean Value Theorem
Suppose that f is differentiable on (a, b) (and the one-sided derivatives exist at a and b).

Note that f MUST be continuous.

The secant line connecting a and b has slope

$$\frac{f(b) - f(a)}{b - a}.$$

Notice that there must be a point where the tangent is parallel to this secant.

Fact (Mean Value Theorem). If f is differentiable on $[a, b]$, then there exists a number c in (a, b) (that is, with $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$