Math 111.01 June 27, 2003
Summer 2003

Calculating Limits Using Limit Laws (2.3)

Example. Suppose that

$$f(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \neq -1, \\ -2, & x = -1, \end{cases}$$

and that $g(x) = x - 1$. Do f and g represent the same function?

Solution: Observe that $\mathscr{D}(g) = \mathbb{R}$ and also $\mathscr{D}(f) = \mathbb{R}$.

If $x \neq -1$, then $f(x) = \frac{x^2-1}{x+1} = x - 1$ so that f, g represent the same function except at $x = -1$. However, $f(-1) = -2 = g(-1)$ so these are actually the same function.

Limit Laws. The following rules make computing limits easier. Suppose that c is a constant and both the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

- $\lim\limits_{x\to a} (f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} (f(x) - g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} (cf(x)) = c \cdot \lim\limits_{x\to a} f(x)$
- $\lim\limits_{x\to a} (f(x) \cdot g(x)) = \left(\lim\limits_{x\to a} f(x)\right) \cdot \left(\lim\limits_{x\to a} g(x)\right)$
- $\lim\limits_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ provided that $\lim_{x\to a} g(x) \neq 0$.
- $\lim\limits_{x\to a} (f(x))^n = \left(\lim\limits_{x\to a} f(x)\right)^n$
- $\lim\limits_{x\to a} c = c$ and $\lim_{x\to a} x = a$

Example. Last class we computed $\lim_{x\to 2}(x^2 + 2x + 1) = 9$. Jusify this.

Solution: Using the limit laws we have:

$$\begin{aligned} \lim_{x\to 2}(x^2 + 2x + 1) &= \lim_{x\to 2} x^2 + \lim_{x\to 2} 2x + \lim_{x\to 2} 1 \\ &= \left(\lim_{x\to 2} x\right)^2 + 2 \cdot \lim_{x\to 2} x + \lim_{x\to 2} 1 \\ &= 2^2 + 2(2) + 1 \\ &= 9 \end{aligned}$$

Example. Use the limit laws to compute $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution:

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\lim\limits_{x\to 0} \sin x}{\lim\limits_{x\to 0} x}$$

Now, $\lim_{x\to 0} \sin x = \sin 0 = 0$ and $\lim_{x\to 0} x = 0$. Thus,

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0} = 1.$$

But this is wrong. In order to use law #5, we need that $\lim_{x\to 0} x \neq 0$; we cannot divide by 0. $\therefore$ We need another method to determine this limit.

Example. Compute $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

Solution: Since $\lim_{x\to -2}(5-3x) = 11$, we can use law #5. Thus,

$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{-8+8-1}{5+6} = -\frac{1}{11}.$$

We can therefore formalize the following property.

Direct Substitution. If f is a polynomial functrion or a rational function, and $a \in \mathscr{D}(f)$, then

$$\lim_{x\to a} f(x) = f(a).$$

Example. Compute

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}.$$

Example. Compute

$$\lim_{h\to 0} \frac{(x+h)^{-1} - x^{-1}}{h}.$$

Example. Compute

$$\lim_{h\to 0} \frac{h}{\sqrt{x+h} - \sqrt{x}}.$$

Fact. If $f(x) \leq g(x)$ for x near a then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} f(x)$$

provided both exist.

Fact (Squeeze Theorem). If $f(x) \leq g(x) \leq h(x)$ for x near a and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

(That is, the value of $\lim_{x\to a} g(x)$ is squeezed and must be L.)

Example. Compute

$$\lim_{x\to 0} x^2 \sin(1/x).$$

Solution: Note that we cannot use law #4 since $\lim_{x\to 0} \sin(1/x)$ DNE.

However, notice that for $x \neq 0$, $-1 \leq \sin(1/x) \leq 1$.

Thus,

$$-x^2 \leq x^2 \sin(1/x) \leq x^2.$$

ie: $f(x) \leq g(x) \leq h(x)$.

Since $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$, the Squeeze Theorem tells us that

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Fact. Suppose $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} e^{f(x)} = e^L$.

Note. The last problem on Assignment #2 uses the above two facts.