

## The Limit of a Function (2.2)

**Example.** A few lectures ago we investigated the function  $f(x) = (1 + x)^{1/x}$ . Notice that  $\mathcal{D}(f) = \{x \neq 0\}$ . We were curious to see what happened as  $x$  approached 0.

$$\begin{aligned}f(0.1) &= 2.5937\dots \\f(-0.1) &= 2.86797\dots \\f(0.01) &= 2.70481\dots \\f(-0.01) &= 2.731999\dots \\f(0.001) &= 2.71692\dots \\f(-0.001) &= 2.71964\dots\end{aligned}$$

We saw that as  $x \rightarrow 0$ ,  $f(x) \rightarrow 2.71828\dots$ , which we call the number  $e$ . We write

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = 2.71828\dots = e.$$

**Example.** Investigate  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  by

- (1) graphing  $f(x) = \frac{\sin x}{x}$  and zooming in on  $(0, 1)$ .
- (2) plugging in  $x = \pm 0.001$ ,  $x = \pm 0.0001$ ,  $x = \pm 0.00001$ .

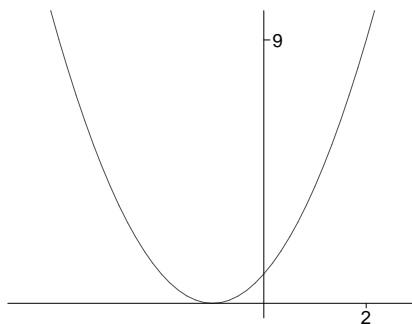
**Definition.** See Stewart page 101: We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of  $f(x)$  as  $x$  approaches  $a$  equals  $L$ ” if we can make the values of  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we like by taking  $x$  sufficiently close to  $a$  (on either side of  $a$ ) but not equal to  $a$ ).

**Example.** Compute

$$\lim_{x \rightarrow 2} (x^2 + 2x + 1).$$



Since the function  $f(x) = x^2 + 2x + 1$  is continuous, and  $\mathcal{D}(f) = \mathbb{R}$ , there is no trouble at  $x = 2$ . Thus, as  $x$  approaches 2,  $x^2 + 2x + 1 \rightarrow 2^2 + 2(2) + 1 = 9$ . That is,

$$\lim_{x \rightarrow 2} (x^2 + 2x + 1) = 9.$$

**Example.** Compute

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}.$$

Let  $f(x) = \frac{x^2 - 4}{x - 2}$ . In this example,  $\mathcal{D}(f) = \{x \neq 2\}$ , so that “ $f(2)$ ” DOES NOT MAKE SENSE. Thus, we cannot consider what happens at  $x = 2$ . However, this is not a problem because the definition of limit only requires us to *approach* 2.

$$\begin{aligned} f(2.01) &= 4.01 \\ f(2.001) &= 4.001 \\ f(2.0001) &= 4.0001 \\ f(1.9999) &= 3.9999 \\ f(1.999) &= 3.999 \\ f(1.99) &= 3.99 \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4.$$

Notice that

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$$

provided that  $x \neq 2$ . Hence, *these do not represent the same function*. In particular,

$$\mathcal{D}\left(\frac{x^2 - 4}{x - 2}\right) = \{x \neq 2\}$$

but

$$\mathcal{D}(x + 2) = \mathbb{R}.$$

However, they are the same away from 2, and since the definition of limit requires us to check what happens as  $x \rightarrow 2$  only, we see that

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \rightarrow 2} x + 2 = 4.$$

**Example.** Compute

(a)

$$\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right),$$

(b)

$$\lim_{x \rightarrow 0} \left( x^3 + \frac{\cos 5x}{10000} \right),$$

(c)

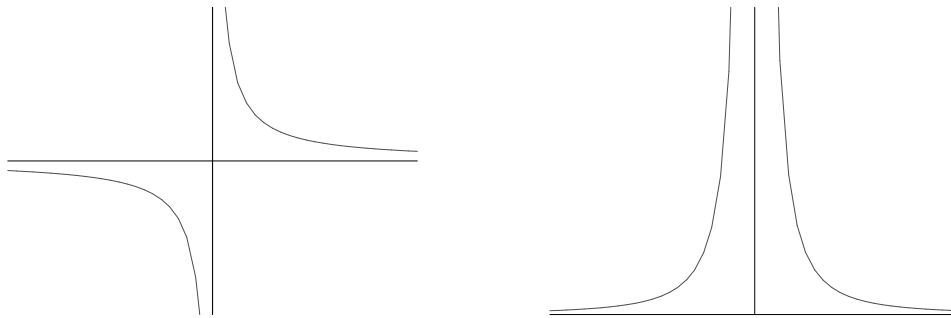
$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}.$$

All of these are in Stewart, and all illustrate the need to be careful using a calculator.

**Example.** Compute

$$\lim_{x \rightarrow 0} \frac{1}{x} \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1}{x^2}$$

by graphing.

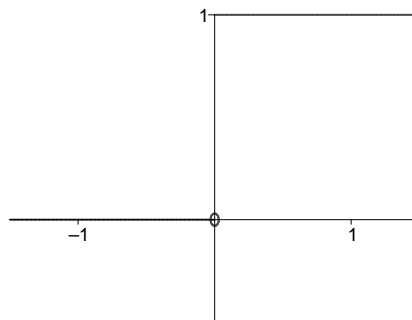


In Section 2.2, both of these limits “DO NOT EXIST.” In Section 2.5, however, we will see a difference.

**Example.** Let  $f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$

(a) Plot  $f(x)$ .

(b) Compute  $\lim_{x \rightarrow 1} f(x)$ ,  $\lim_{x \rightarrow -1} f(x)$ , and  $\lim_{x \rightarrow 0} f(x)$ .



Clearly we see that  $\lim_{x \rightarrow 1} f(x) = 1$  and  $\lim_{x \rightarrow -1} f(x) = 0$ .

However,  $\lim_{x \rightarrow 0} f(x)$  DOES NOT EXIST.

If  $x > 0$ , then  $f(x) \rightarrow 1$ , but if  $x < 0$ , then  $f(x) \rightarrow 0$ .

In order to deal with such a situation, we have *one-sided limits*.

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad (\text{limit from the right})$$

$$\lim_{x \rightarrow 0^-} f(x) = 0 \quad (\text{limit from the left})$$

**Definition.** The limit  $\lim_{x \rightarrow a} f(x) = L$  if and only if both  $\lim_{x \rightarrow a^+} f(x) = L$  and  $\lim_{x \rightarrow a^-} f(x) = L$ . That is, if the limit from the left exists, and the limit from the right exists, and they both have the same value, then the limit itself is defined to be this common value.