Functions, Derivatives, and Other Loose Ends

Math 111 Section 01

July 3, 2003

1 Functions

Definition. If f is a function represented by $f(x)$, then its *graph* is the set of points

$$\{(x, f(x)) : x \in \mathbb{R}\} = \{(x, y) : y = f(x), x \in \mathbb{R}\}.$$

Vertical Line Test: A curve in the (x, y)-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Definition. An *even function* satisfies $f(-x) = f(x)$, while an *odd function* satisfies $f(-x) = -f(x)$.

Example 1. $f(x) = \sin x$ and $g(x) = x^3$ are both odd functions, while $h(x) = \cos x$ and $j(x) = x^2$ are both even functions. The functions $k(x) = \cos x + \sin x$ and $\ell(x) = x^3 + 1$, say, are neither even nor odd functions.

Definition. A function is *one-to-one* if no horizontal line intersects its graph more than once.

Fact 1. One-to-one functions have well-defined inverses.

If f is one-to-one with domain $\mathscr{D}(f) = A$ and range $\mathscr{R}(f) = B$, then its inverse function f^{-1} has domain $\mathscr{D}(f^{-1}) = B$ and range $\mathscr{R}(f^{-1}) = A$.

$$f^{-1}(y) = x \;\Leftrightarrow\; y = f(x)$$

Example 2. Suppose that f is given by $f(x) = \frac{x}{x+1}$. Then $\mathscr{D}(f) = \{x \neq -1\}$ and graphically we see that f is one-to-one. If we write $y = \frac{x}{x+1}$ and solve for x, then algorithmically we will find f^{-1}. Thus,

$$y = \frac{x}{x+1} \;\Rightarrow\; yx + y = x \;\Rightarrow\; x(y-1) = -y \;\Rightarrow\; x = \frac{y}{1-y}.$$

Hence $f^{-1}(x) = \frac{x}{1-x}$, and $\mathscr{D}(f^{-1}) = \mathscr{R}(f) = \{x \neq 1\}$. Note that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

2 The Derivative

Recall the definition of derivative. If $f(x)$ is a function, then

- $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
- $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
- $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Questions. What does each of these mean? Are they different definitions, or are they really just saying the same thing? Why? When should you use one instead of the others?

Definition. A function is *differentiable at a* if $f'(a)$ exists.

Definition. A function is *differentiable* if $f'(a)$ exists for every $a \in \mathcal{D}(f)$.

Example 3. If $f(x) = \dfrac{1-x}{(2+x)^2}$ show that $f'(x) = \dfrac{-3}{(2+x)^3}$. What are the domains of f and f'? See Stewart page 161.

Notation. If $y = f(x)$, then

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

Fact 2. If a function is continuous, then it is not necessarily differentiable. For example, $f(x) = |x|$ is not differentiable at 0, although it is differentiable everywhere else.

Fact 3. If a function is continuous, then it must be differentiable.

These do not say the same thing!

Sometimes we cannot compute the derivative at a point exactly, and instead we need to approximate the derivative by secant lines.

Example 4. See Example 2 Page 159 in Stewart. Notice the use of $\approx$ NOT $=$, and that it is necessary to approximate from both sides.

3 How can a function fail to be differentiable?

- It can have a "sharp corner." *Example:* $f(x) = |x|$ at $x = 0$.
- It can be discontinuous. *Example:* $g(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is discontinuous at $x = 0$.
- It can have "infinite" slope. *Example:* $h(x) = x^{2/3}$ at $x = 0$.

Check that these three functions are truly non-differentiable. Notice that f and h are continuous.

4 The Second Derivative

The second derivative is just the derivative of the the (first) derivative, and is denoted $f''(x) = \dfrac{d^2}{dx^2}f(x)$. Hence,

- $f''(a) = \lim\limits_{x \to a} \dfrac{f'(x) - f'(a)}{x - a}$
- $f''(a) = \lim\limits_{h \to 0} \dfrac{f'(a+h) - f'(a)}{h}$
- $f''(x) = \lim\limits_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h}$

Example 5. Show that if $f(x) = x^3$, then $f''(x) = 6x$.

5 Distance, Velocity, Acceleration

Suppose that the function $s(t)$ denotes the **position** of an object at time t. (Most likely, $0 \le t < \infty$). If you imagine the position as a point on the real line, then $s(t)$ denotes the **displacement** from the origin and could very easily be negative.

[However, the **distance** from the origin is always positive and is given by $d(t) = |s(t)|$.]

The **average velocity** of the object over the time interval $[t_0, t_0 + h]$ is

$$\text{average velocity} = \frac{\text{net displacement}}{\text{time}} = \frac{s(t_0 + h) - s(t_0)}{h}$$

Notice that this is just the slope of the secant line over $[t_0, t_0 + h]$ on the graph of $s(t)$.

We define the **(instantaneous) velocity** as

$$\text{velocity at } t_0 = v(t_0) = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Thus, velocity is the derivative of position: $v(t) = s'(t)$.

However, in many problems we cannot compute the velocity exactly, and must instead approximate it by the average velocity.

The **average acceleration** of the object over the time interval $[t_0, t_0 + h]$ is

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time}} = \frac{v(t_0 + h) - v(t_0)}{h}$$

Notice that this is just the slope of the secant line over $[t_0, t_0 + h]$ on the graph of $v(t)$.

We define the **(instantaneous) acceleration** as

$$\text{acceleration at } t_0 = a(t_0) = \lim_{h \to 0} \frac{v(t_0 + h) - v(t_0)}{h}$$

Thus, acceleration is the derivative of velocity and the second derivative of position: $a(t) = v'(t) = s''(t)$.

As noted on page 167 of Stewart, the **jerk** is the instantaneous rate of change of acceleration, or $j(t) = a'(t) = v''(t) = s'''(t)$. [A large jerk means a sudden change in acceleration.]

Example 6 (Exercise #15 Page 148). If a ball is thrown into the air with a velocity of 40 ft/s, its height (in ft) after t seconds is given by $h = 40t - 16t^2$. Find the velocity when $t = 2$.

Solution:

$$v(2) = \lim_{t \to 2} \frac{h(t) - h(2)}{t - 2} = \lim_{t \to 2} \frac{(40t - 16t^2) - (80 - 64)}{t - 2} = \cdots .$$

6 Asymptotes

To determine horizontal asymptotes of $f(x)$, we must investigate the behaviour as $x \to \pm\infty$.

Example 7. Determine the horizontal asymptotes of $f(x) = \dfrac{3x^2 + 4}{4 - x^2}$.

Solution: We simply compute.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2+4}{4-x^2} = \lim_{x\to\infty} \frac{3+4/x^2}{4/x^2-1} = \frac{3-0}{0-1} = -3.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x^2+4}{4-x^2} = \lim_{x\to-\infty} \frac{3+4/x^2}{4/x^2-1} = \frac{3-0}{0-1} = -3.$$

Thus the line $y = -3$ is a (two-sided) horizontal asymptote.

To determine vertical asymptotes of $f(x)$, we must determine where $f(x) \to \pm\infty$. This most likely occurs at points not in the domain of f.

Example 8. Determine the vertical asymptotes of $f(x) = \dfrac{3x^2+4}{4-x^2}$.

Solution: Since $f(x) = \dfrac{3x^2+4}{4-x^2} = \dfrac{3x^2+4}{(2-x)(2+x)}$, there are two points not in $\mathcal{D}(f)$, namely 2 and -2.

We compute the one sided limits, beginning with $x = 2$.

$$\lim_{x\to 2^+} \frac{3x^2+4}{(2-x)(2+x)} = -\infty.$$

$$\lim_{x\to 2^-} \frac{3x^2+4}{(2-x)(2+x)} = \infty.$$

You should check these.

Thus the line $x = 2$ is a vertical asymptote.

We now compute the one sided limits for $x = -2$.

$$\lim_{x\to -2^+} \frac{3x^2+4}{(2-x)(2+x)} = \infty.$$

$$\lim_{x\to -2^-} \frac{3x^2+4}{(2-x)(2+x)} = -\infty.$$

You should check these, too.

Thus the line $x = -2$ is a vertical asymptote.

Example 9. Use information about the asymptotes, and the function itself, to sketch a graph of f, without using a calculator.

7 Other Wild Functions

Here are two interesting functions whose definitions are not entirely given via formulae, but rather through discription.

Example 10 (Greatest Integer Function). Let $[\![x]\!]$ denote the greatest integer less than or equal to x.

Thus, $[\![x]\!]$ is only integer-valued.

For example, $[\![2.5]\!] = 2$, $[\![5.9999987]\!] = 5$, $[\![1]\!] = 1$, $[\![1/\pi]\!] = 0$, $[\![-3.432]\!] = -4$, and $[\![-6.5]\!] = -7$.

Draw a graph of $f(x) = [\![x]\!]$. Where is f continuous? not continuous? differentiable? not differentiable?

Example 11. Suppose that $I(x) = \begin{cases} 1, & x \text{ is rational}, \\ 0, & x \text{ is irrational}. \end{cases}$

Show that $I(x)$ is not continuous at $x = 0$. You will need to argue why the definition of continuity fails at 0, as opposed to "actually computing."

In fact, your argument shows that $I(x)$ is *not* continuous at *any* point. It is nowhere continuous!