Math 111.01 Summer 2003
Assignment #6 Solutions

1. Practice problems.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.

2. Extra practice.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.
Section 5.3 #12 Answer: $20/3$
Section 5.3 #14 Answer: 0
Section 5.3 #16 Answer: $7/10$
Section 5.3 #18 Answer: 2
Section 5.3 #20 Answer: $86/7$
Section 5.3 #22 Answer: $3/2 + \ln 2$
Section 5.3 #24 Answer: $2e^5 + 4\sin 5 - 2 \approx 290.99$
Section 5.3 #26 Answer: $33/4$
Section 5.3 #30 Answer: $-1 + 2/3 \cdot \sqrt{3}$

3. Problems to hand in.

Section 4.9

#6. Since $f(x) = \sqrt[3]{x^2} - \sqrt{x^3} = x^{2/3} - x^{3/2}$, the most general antiderivative is

$$F(x) = \frac{1}{5/3}x^{5/3} - \frac{1}{5/2}x^{5/2} + C = \frac{3}{5}x^{5/3} - \frac{2}{5}x^{5/2} + C.$$

#10. Since $f(x) = 3e^x + 7\sec^2 x$, the most general antiderivative is

$$F(x) = 3e^x + 7\tan x + C_n, \text{ on the interval } (n\pi - \tfrac{\pi}{2}, n\pi + \tfrac{\pi}{2}).$$

Notice that in this case, there are many different "pieces" in the antiderivative. Each "piece" may have a different constant and these will all be different antiderivative functions.

#12. Since $f(x) = \frac{x^2+x+1}{x} = x + 1 + x^{-1}$, the most general antiderivative is

$$F(x) = \frac{1}{2}x^2 + x + \ln x + C.$$

#20. If $f'(x) = 4/\sqrt{1-x^2}$ then,

$$f(x) = 4\arcsin x + C$$

$$f(\frac{1}{2}) = 4\arcsin(\frac{1}{2}) + C = 4(\frac{\pi}{6}) + C.$$

Since $f(\frac{1}{2}) = 1$, we have

$$\frac{2\pi}{3} + C = 1 \Rightarrow C = 1 - \frac{2\pi}{3},$$

so $f(x) = 4\arcsin x + 1 - \frac{2\pi}{3}$.

#24.

$$f''(x) = 3e^x + 5\sin x \Rightarrow f'(x) = 3e^x - 5\cos x + C$$

and since $f'(0) = 2$,

$$3 - 5 + C = 2 \Rightarrow C = 4$$

Therefore, $f'(x) = 3e^x - 5\cos x + 4$. Next, we take another antiderivative.

$$f(x) = 3e^x - 5\sin x + 4x + D$$

Since $f(0) = 1$, $3 + D = 1 \Rightarrow D = -2$. This means that $f(x) = 3e^x - 5\sin x + 4x - 2$.

#36.

$$a(t) = v'(t) = 5 + 4t - 2t^2 \Rightarrow v(t) = 5t + 2t^2 - \frac{2}{3}t^3 + C.$$

$$v(0) = 5(0) + 2(0)^2 - \frac{2}{3}(0^3) + C = 3 \Rightarrow C = 3$$

so,

$$v(t) = s'(t) = 5t + 2t^3 - \frac{2}{3}t^3 + 3 \Rightarrow s(t) = \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 + 3t + D$$

$$s(0) = 0 + 0 + 0 + 0 + D = 10 \Rightarrow D = 10$$

Therefore, the particle's position after t seconds is given by $s(t) = \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 + 3t + 10$.

#40. Consider first Example 7 from the text and see that the distance function for the first ball can be written as $s_1(t) = -16t^2 + 48t + 432$. For the second ball, we find the equation for the position function by antidifferentiation.

$$\begin{aligned} a_2(t) &= -32 \\ \Rightarrow \quad v_2(t) &= -32t + C \\ \\ \text{So,} \quad v_2(1) &= -32(1) + C = 24 \\ \Rightarrow \quad C &= 56 \\ v_2(t) &= -32t + 56 \\ \\ s_2(t) &= -16t^2 + 56t + D \\ \text{So,} \quad s_2(1) &= -16(1)^2 + 56(1) + D = 432 \\ \Rightarrow \quad D &= 392 \\ s_2(t) &= -16t^2 + 56t + 392 \end{aligned}$$

When the balls pass each other, $s_1(t) = s_2(t)$. We set these expressions equal and solve for t to find the time they are passing each other:

$$-16t^2 + 48t + 432 = -16t^2 + 56t + 392 \iff 8t = 40 \iff t = 5 \text{ seconds.}$$

Section 5.1

#2. See problem in text for graph of the function.

a. Find the area from $x = 0$ to $x = 12$ by splitting into 6 subintervals: each of the 6 subintervals has width $12/6 = 2$.

(i) Using left endpoints: $L_6 = 2\times 9 + 2\times 8.75 + 2\times 8.25 + 2\times 7.3 + 2\times 6 + 2\times 4 = 86.6$.

(ii) Using right endpoints: $R_6 = 2\times 8.75+2\times 8.25+2\times 7.3+2\times 6+2\times 4+2\times 1 = 70.6.$

(iii) Using midpoints: $M_6 = 2\times 8.9+2\times 8.5+2\times 7.8+2\times 6.7+2\times 5+2\times 2.9 = 79.6.$

b. Since f is decreasing, L_6 is an overestimate.

c. Since f is decreasing, R_6 is an underestimate.

d. The best estimate is given by M_6 since, in this case, the area of each rectangle is closer to the actual area than in the case of the overestimates and understimates in (i) and (ii).

#12. The height of *Endeavour* 62 seconds after liftoff is given by:

$$\begin{aligned} R_6 &= v(10)(10-0) + v(15)(15-10) + v(20)(20-15) + v(32)(32-20) \\ &\qquad + v(59)(59-32) + v(62)(62-59) \\ &= 185\times 10 + 319\times 5 + 447\times 5 + 742\times 12 + 1325\times 27 + 1445\times 3 \\ &= 54694, \end{aligned}$$

$$\begin{aligned} L_6 &= v(0)(10-0) + v(10)(15-10) + v(15)(20-15) + v(20)(32-20) \\ &\qquad + v(32)(59-32) + v(59)(62-59) \\ &= 0\times 10 + 185\times 5 + 319\times 5 + 447\times 12 + 742\times 27 + 1325\times 3 \\ &= 31893. \end{aligned}$$

Averaging the two estimates, we get 43 293.5 feet.

#14. We can use right endpoints (and get an overestimate, why?) or left endpoints (and get an underestimate, why?). Let's then use midpoints this time. Split the time interval into 6 equal pieces of 5 seconds, that is, 5/3600 hours.

$$\begin{aligned} M_6 &= \frac{5}{3600}\times[v(2.5)+v(7.5)+v(12.5)+v(17.5)+v(22.5)+v(27.5)] \\ &= \frac{5}{3600}\times[31.25+66+88+103.5+113.75+119.25] \\ &= \frac{5}{3600}(521.75) = 0.725 \text{ km}. \end{aligned}$$

#16. The area under $f(x) = \frac{\ln x}{x}$ from $x = 3$ to $x = 10$ is given by the formula

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

where $\Delta x = \frac{10-3}{n}$, and $x_i = 3 + \frac{i\cdot 7}{n}$, for $i = 1, 2, \ldots, n$. Therefore, we get

$$A = \lim_{n\to\infty}\left[\frac{\ln(3+\frac{7}{n})}{3+\frac{7}{n}}\cdot\frac{7}{n} + \frac{\ln(3+\frac{2\cdot 7}{n})}{3+\frac{2\cdot 7}{n}}\cdot\frac{7}{n} + \cdots + \frac{\ln(10)}{10}\cdot\frac{7}{n}\right] = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{7\ln(3+\frac{7i}{n})}{3n+7i}.$$

Section 5.2

#6. **a**. Using the right endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(x_i)\,\Delta x &= 1[g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)]\\ &\approx 1-0.5-1.5-1.5-0.5+2.5=-0.5.\end{aligned}$$

b. Using the left endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(x_{i-1})\,\Delta x = \sum_{i=0}^{5} g(x_i)\,\Delta x &= 1[g(-3)+g(-2)+g(-1)+g(0)+g(1)+g(2)]\\ &\approx 2+1-0.5-1.5-1.5-0.5=-1.\end{aligned}$$

c. Using the midpoints of each subinterval to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(\overline{x}_i)\,\Delta x &= \sum_{i=1}^{6} g\left(\frac{x_i+x_{i-1}}{2}\right)\Delta x\\ &= 1[g(-2.5)+g(-1.5)+g(-0.5)+g(0.5)+g(1.5)+g(2.5)]\\ &\approx 1.5+0-1-1.75-1+0.5=-1.75.\end{aligned}$$

#18. On $[1,5]$,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{e^{x_i}}{1+x_i}\Delta x=\int_1^5\frac{e^x}{1+x}\,dx.$$

#24. Letting $\Delta x = 5/n$ and $x_i = 5i/n$, we have

$$\begin{aligned}\int_0^5 (1+2x^3)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+2\left(\frac{5i}{n}\right)^3\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty}\left(\frac{5}{n}\right)\sum_{i=1}^{n}\left(1+2\cdot\frac{125i^3}{n^3}\right)\\ &= \lim_{n\to\infty}\left(\frac{5}{n}\right)\left(1\cdot n+2\cdot\frac{125}{n^3}\sum_{i=1}^{n} i^3\right) = \lim_{n\to\infty}\left(\frac{5}{n}\right)\left(1\cdot n+2\cdot\frac{125}{n^3}\frac{(n+1)^2}{n^2}\right)\\ &= \lim_{n\to\infty}\left(5+312.5\left(1+\frac{1}{n}\right)^2\right) = 5+312.5=317.5\end{aligned}$$

#36. $\int_0^3 |3x-5|\,dx$ can be interpreted as the area under the graph of the function $f(x)=|3x-5|$ between $x=0$ and $x=3$. This is equal to the area of the two triangles, so

$$\int_0^3 |3x-5|\,dx = \frac{1}{2}\cdot\frac{5}{3}\cdot 5+\frac{1}{2}\cdot\left(3-\frac{5}{3}\right)\cdot 4=\frac{41}{6}.$$

#42. By rule 5 on page 365, $\int_0^4 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt - \int_3^4 f(t)\,dt$. Thus,

$$\int_1^3 f(t)\,dt = \int_0^4 f(t)\,dt - \int_3^4 f(t)\,dt - \int_0^1 f(t)\,dt = (-6) - (1) - (2) = -9.$$

#46. Since $m \le f(x) \le M$ for all x, by property 8 on page 366, we have

$$2m \le \int_0^2 f(x)\,dx \le 2M.$$

Section 5.3

#6. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Total Change Theorem, $\int_3^5 f(x)\,dx = \int_3^5 E'(x)\,dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.

#10. Since $-x^{-1}$ is an antiderivative of x^{-2}, by the Evaluation Theorem we have

$$\int_1^2 x^{-2}\,dx = -x^{-1}\Big|_1^2 = \left(-2^{-1}\right) - \left(-1^{-1}\right) = \frac{1}{2}.$$

#28. Since e^x is an antiderivative of e^x, by the Evaluation Theorem we have

$$\int_{\ln 3}^{\ln 6} 8e^x\,dx = 8e^x\Big|_{\ln 3}^{\ln 6} = 8e^{\ln 6} - 8e^{\ln 3} = 48 - 24 = 24.$$

#32. Since $\arcsin x = \sin^{-1} x$ is an antiderivative of $\frac{1}{\sqrt{1-x^2}}$, by the Evaluation Theorem we have

$$\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{0.5} = \sin^{-1} 0.5 - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

#34. Since the absolute value of a is defined by $|a| = \begin{cases} a, & \text{if } a \ge 0, \\ -a, & \text{if } a < 0, \end{cases}$ we must determine where $x - x^2$ is both positive and negative. Since $x - x^2 = x(1-x)$, we see that if $x > 1$ it is negative, if $x < 0$ it is also negative, and if $0 < x < 1$ it is positive. We now apply property 5 (page 365) of definite integrals in order to handle the absolute value, splitting up at the places where $x - x^2$ changes from positive to negative. Hence,

$$\begin{aligned}\int_{-1}^2 |x - x^2|\,dx &= \int_{-1}^0 |x - x^2|\,dx + \int_0^1 |x - x^2|\,dx + \int_1^2 |x - x^2|\,dx \\ &= \int_{-1}^0 -(x - x^2)\,dx + \int_0^1 (x - x^2)\,dx + \int_1^2 -(x - x^2)\,dx.\end{aligned}$$

Since $\frac{1}{2}x^2 - \frac{1}{3}x^3$ is an antiderivative of $x - x^2$, by the Evaluation Theorem we have

$$\begin{aligned}
&\int_{-1}^{0} (x - x^2)\ dx + \int_{0}^{1} -(x - x^2)\ dx + \int_{1}^{2} (x - x^2)\ dx \\
&\quad = -\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{-1}^{0} + \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{0}^{1} + -\left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{1}^{2} \\
&\quad = \frac{7}{3} + \frac{3}{2} - 2 \\
&\quad = \frac{11}{6}.
\end{aligned}$$

#52. As on page 375, the displacement is given by

$$\begin{aligned}
\int_1^6 v(t)\ dt = \int_1^6 t^2 - 2t - 8\ dt = \left(\frac{1}{3}t^3 - t^2 - 8t\right)\Big|_1^6 &= \left(\frac{1}{3}6^3 - 6^2 - 8\cdot 6\right) - \left(\frac{1}{3}1^3 - 1^2 - 8\cdot 1\right) \\
&= (72 - 36 - 48) - \left(\frac{1}{3} - 1 - 8\right) \\
&= -\frac{10}{3}.
\end{aligned}$$

Thus the displacement during the time interval $[1, 6]$ was $-10/3$ m.

As on page 375, the total distance travelled is given by $\int_1^6 |v(t)|\ dt = \int_1^6 |t^2 - 2t - 8|\ dt$. As in problem #34, we need to consider the cases where this is positive and where this is negative. Since $t^2 - 2t - 8 = (t - 4)(t + 2)$, we see that for $1 \leq t \leq 4$, $t^2 - 2t - 8$ is negative, while for $4 \leq t \leq 6$, $t^2 - 2t - 8$ is positive. Thus,

$$\begin{aligned}
\int_1^6 |t^2 - 2t - 8|\ dt &= \int_1^4 -(t^2 - 2t - 8)\ dt + \int_4^6 t^2 - 2t - 8\ dt \\
&= -\left(\frac{1}{3}t^3 - t^2 - 8t\right)\Big|_1^4 + \left(\frac{1}{3}t^3 - t^2 - 8t\right)\Big|_4^6 \\
&= \frac{98}{3}.
\end{aligned}$$

Therefore, the total distance travelled during the time interval $[1, 6]$ was $98/3$ m.

Section 5.4

#6. The function $g(x)$ represents the area in the t, y-plane under the curve $y = 2 + \cos(t)$, above the t-axis, between the vertical lines $t = \pi$ and $t = x$. (Note that x is used for something else here, other than the usual dependent variable in cartesian coordinates.)

- **a**. By FTC 1, $g(x) = \int_\pi^x (2 + \cos t)\ dt$, so $g'(x) = f(x) = (2 + \cos x)$.
- **b**. By FTC 2, $g(x) = \int_\pi^x (2 + \cos t)\ dt = (2t + \sin t)\Big|_\pi^x = 2x + \sin x - 2\pi$. Thus, $g'(x) = 2 + \cos x$.

#12. Let $u = x^2$. Then $\frac{du}{dx} = 2x$, and $\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$, by the Chain Rule. So

$$\frac{dh}{dx} = \frac{d}{dx}\int_0^{x^2} \sqrt{1+r^3}\,dr = \frac{d}{du}\int_0^{u} \sqrt{1+r^3}\,dr \cdot \frac{du}{dx} = \sqrt{1+u^3}(2x) = 2x\sqrt{1+x^6}.$$

#14. Similarly to #12, let $u = e^x$. Then $\frac{du}{dx} = e^x$, and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, by the Chain Rule. So

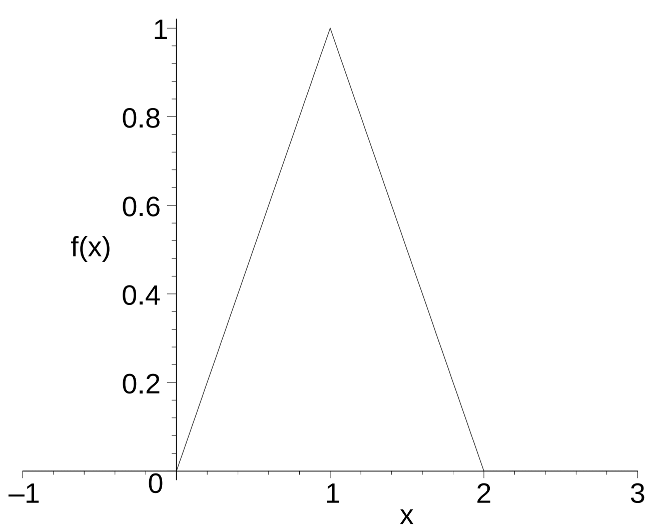
$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\int_{e^x}^{0} \sin^3(t)\,dt = \frac{d}{du}\int_{u}^{0} \sin^3(t)\,dt \cdot \frac{du}{dx} \\ &= -\frac{d}{du}\int_0^{u} \sin^3(t)\,dt \cdot \frac{du}{dx} = -\sin^3(u)(e^x) = -e^x\sin^3(e^x).\end{aligned}$$

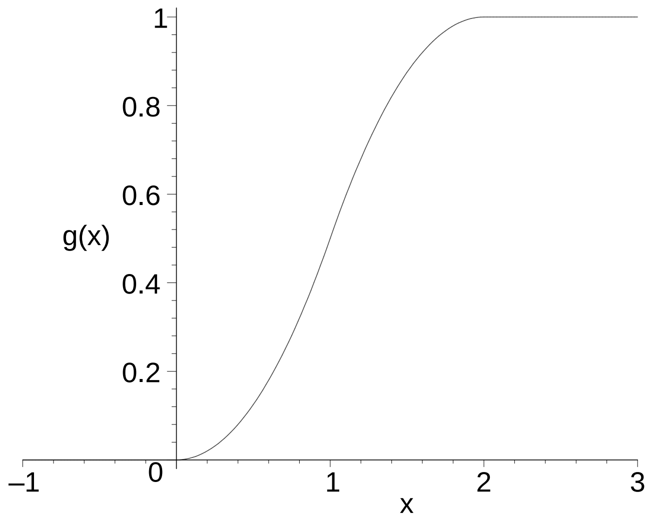
#18. For the curve to be concave up, we need $y'' > 0$. $y(x) = \int_o^x \frac{1}{1+t+t^2}\,dt$. So, by FTC 1, $y'(x) = \frac{1}{1+x+x^2}$, and $y''(x) = -\frac{1+2x}{(1+x+x^2)^2}$. For $y''(x)$ to be positive, we need $-(1+2x)$ to be positive (since the denominator is positive for all x). this occurs only when $x < -1/2$. Thus, $y(x)$ is convcave upward on $(-\infty, -1/2)$.

#24. **a**. If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$. If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left(\frac{1}{2}t^2\right)\Big|_0^x = \frac{1}{2}x^2$. If $1 < x < 2$, then $g(x) = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = g(1) + \int_1^x (2-t)\,dt = \frac{1}{2} + \left(2t - \frac{1}{2}t^2\right)\Big|_0^x = 2x - \frac{1}{2}x^2 - 1$. If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^x f(t)\,dt = g(2) + \int_2^x 0\,dt = g(2) = 1$. Therefore,

$$g(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}x^2, & \text{if } 0 \le x \le 1, \\ 2x - \frac{1}{2}x^2 - 1, & \text{if } 1 < x \le 2, \\ 1, & \text{if } x > 2. \end{cases}$$

b. The graphs of f and g are shown below.





c. f is not differentiable at its corners, namely at $x = 0$, $x = 1$, and $x = 2$. f is differentiable on $(-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$. However, g is differentiable everywhere; you just need to check $g'(x)$ at $x = 0$, 1, 2. Notice that this is merely the same thing as checking the continuity of f at $x = 0$, $x = 1$, and $x = 2$. f is continuous at all of those points, so g is differentiable there.