Math 111.01 Summer 2003
Assignment #5 Solutions

1. I hope you did!

2. Practice problems.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.

3. Problems to hand in.

Section 4.5

#4. **a**. Type $\left[0^0\right]$ indeterminate form.

b. $\lim_{x\to a}[f(x)]^{p(x)} = 0$.

c. Type $[1^\infty]$ indeterminate form.

d. Type $\left[\infty^0\right]$ indeterminate form.

e. $\lim_{x\to a}[p(x)]^{q(x)} = \infty$.

f. $\lim_{x\to a} \sqrt[q(x)]{p(x)} = \lim_{x\to a}[p(x)]^{1/q(x)}$, so this is a type $\left[\infty^0\right]$ indeterminate form.

#10. $\lim_{x\to\pi} \frac{\tan x}{x} = \frac{0}{\pi} = 0$, by continuity. (L'Hôpital's rule does NOT apply here!)

#14. $\lim_{x\to\infty} \frac{e^x}{x^3}$ is an indeterminate form of type $\left[\frac{0}{0}\right]$, so we may apply L'Hôpital's rule. In fact, we must apply it three times:

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty.$$

#34. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$ is a $[1^\infty]$ type indeterminate form. We thus note that $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{\lim_{x\to\infty} bx\ln(1+\frac{a}{x})}$. But $\lim_{x\to\infty} bx\ln(1+\frac{a}{x})$ is a $[\infty\cdot 0]$ type indeterminate form, so we compute

$$\lim_{x\to\infty} bx\ln(1+\frac{a}{x}) = b\lim_{x\to\infty}\frac{\ln(1+\frac{a}{x})}{1/x} = b\lim_{x\to\infty}\frac{\frac{1}{1+\frac{a}{x}}\cdot\frac{-a}{x^2}}{-1/x^2} = b\lim_{x\to\infty}\frac{a}{1+\frac{a}{x}} = ab.$$

Thus,

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}.$$

#44. Note first that xe^{-x^2} has no vertical asymptotes because it is continuous for all x. To find horizontal asymptotes, we must compute $\lim_{x\to\infty} xe^{-x^2}$ and $\lim_{x\to-\infty} xe^{-x^2}$. These are both $[\infty\cdot 0]$ forms. Using L'Hôpital's rule, we find that

$$\lim_{x\to\pm\infty} xe^{-x^2} = \lim_{x\to\pm\infty}\frac{x}{e^{x^2}} = \lim_{x\to\pm\infty}\frac{1}{2xe^{-x^2}} = 0.$$

Thus the line $y = 0$ is the horizontal asymptote.

#54. **a**. $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}(1 - e^{-ct/m}) = \frac{mg}{c}$, since $\lim_{t\to\infty} -ct/m = -\infty$. This means that the limiting velocity as time goes on is $\frac{mg}{c}$.

b. $\lim_{m\to\infty} v = \lim_{m\to\infty} \frac{mg}{c}(1 - e^{-ct/m}) = \frac{mg}{c}$ is a $[\infty \cdot 0]$ type indeterminate form. We use L'Hôpital's rule to compute $\lim_{m\to\infty} \frac{mg}{c}(1 - e^{-ct/m}) = \frac{g}{c} \lim_{m\to\infty} \frac{(1-e^{-ct/m})}{1/m}$ $= \frac{g}{c} \lim_{m\to\infty} \frac{-ct/m^2 e^{-ct/m}}{-1/m^2} = gt \lim_{m\to\infty} e^{-ct/m} = gt$. Thus for very heavy objects, the velocity increases approximately linearly with time and is not dependent on mass.

Section 4.6

#4. Let x be any number in $(0, \infty)$, i.e., x is any positive number. The sum of x and its reciprocal is $f(x) = x + \frac{1}{x}$, so we seek to minimize f. We will have to apply the first derivative test for absolute minima. We first compute $f'(x) = 1 - \frac{1}{x^2}$. Setting $f'(x) = 0$, we find that $1 - \frac{1}{x^2} = 0$, or $1 = \frac{1}{x^2}$, or $x = 1$. (We can ignore the root $x = -1$ since we only care about positive values of x.) We easily find that $f'(x) < 0$ for all $0 < x < 1$, so f is decreasing for $0 < x < 1$. Since $f'(x) > 0$ for $x > 1$, f is increasing for $x > 1$. By the first derivative test, $f(1)$ is thus an absolute minimum of f on $(0, \infty)$, and 1 is thus our desired positive number.

#8. **a**. Each of the boxes in Figure 1 can be constructed by cutting four $z \times z$ corners out of a 3×3 piece of cardboard. Box 1 has volume $2 \times 2 \times .5 = 2$, Box 2 has volume $1 \times 1 \times 1 = 1$, and Box 3 has volume $.5 \times .5 \times 2 = .5$. It appears that the maximum volume will be achieved by taking a short, flat box, so we try a few more possibilities: $1.5 \times 1.5 \times .75 = 1.68...$; $2.5 \times 2.5 \times .25 = 1.56...$; $2.25 \times 2.25 \times .375 = 1.89....$ Thus we guess that the maximum volume will be around 2.

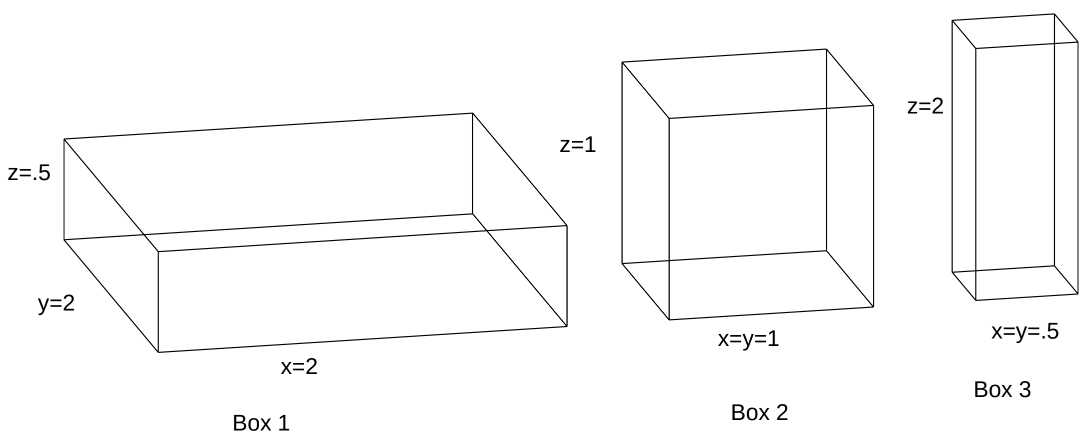


Figure 1: figure for #8 a

b. See Figure 2 on the following page.

c. $V(x, z) = x^2 z$.

d. $x + 2z = 3$, so $x = 3 - 2z$.

e. $V(z) = (3 - 2z)^2 z = (9 - 12z + 4z^2)z = 9z - 12z^2 + 4z^3$.

f. We first note that $0 \le z \le 1.5$, so we shall use the closed interval method to find the minimum of V on this interval. $V'(z) = 9 - 24z + 12z^2 = 3(4z^2 - 8z + 3) =$

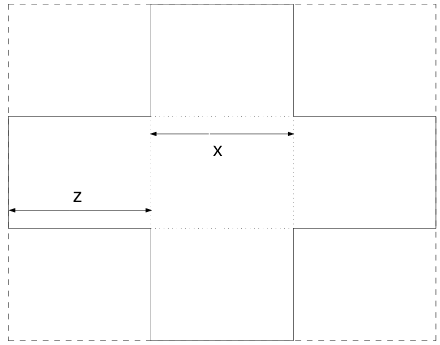


Figure 2: figure for #8 b

$3(2z-3)(2z-1)$, which has zeros at $z=1/2$ and $z=3/2$. We thus compute $V(0)=0$, $V(1/2)=2^2\cdot .5=2$, and $V(3/2)=0$. By the closed interval method, the maximum volume is $V(1/2)=2$ ft^3.

#10. Let x represent the length and width of the square base of the box, and let z be its height as in the previous problem. The volume is 32,000 cm$^3=x^2z$, and the amount of material used is the surface area, that is, $S(x,z)=x^2+4xz$. Using the volume constraint, we find that $z=\frac{32000}{x^2}$, so we may eliminate z in S to find $S(x)=x^2+\frac{128{,}000}{x}$. We also note that x may be any positive number, so we will use the first derivative test for absolute maxima and minima to minimize S. $S'(x)=2x-\frac{128{,}000}{x^2}$. Setting $S(x)=0$, we find that $2x=\frac{128{,}000}{x^2}$, or $x^3=64,000$ or $x=40$ cm. We see that $S'(x)<0$ for $x<40$ and $S'(x)>0$ for $x>40$, so that $S(40)$ is an absolute minimum by the first derivative test for absolute extrema. To complete the problem, we find $z=\frac{32000}{40^2}=20$ cm, so the base should be 40 cm on each side and the height 20 cm.

#12. Let x be the length of the shorter side of the base, so the other side of the base has length $2x$. Also, let z be the height of the box. The volume is given by $2x\cdot x\cdot z=10$ m^3. Also, to get the cost, we add 10 times the surface area of the bottom and 6 times the total surface area of the sides. That is, $C(x,z)=10(2x\cdot x)+6(2\cdot 2xz+2\cdot xz)=20x^2+36xz$. We then solve for z in our volume constraint to find that $z=\frac{5}{x^2}$. Substituting into the cost equation, we find that $C(x)=20x^2+\frac{180}{x}$. x may be any positive number here, so we will use the first derivative test for absolute extrema. $C'(x)=40x-\frac{180}{x^2}$. Setting $C=0$, we find that $40x=\frac{180}{x^2}$, or $x^3=\frac{180}{40}=\frac{9}{2}$. Thus $x=\sqrt[3]{\frac{9}{2}}$ is a critical point of C. We see that $C'(x)<0$ if $x<\sqrt[3]{\frac{9}{2}}$ and $C'(x)>0$ if $x>\sqrt[3]{\frac{9}{2}}$, so the absolute minimum of $C(x)$ is about \$$\sqrt[3]{\frac{9}{2}}$ =\$163.54 by the first derivative test.

#16. See Figure 3 on the following page for a diagram. The rectangle displayed has an area of $A(x,y)=2xy$. Since $y=8-x^2$, we substitute to find $A(x)=2x(8-x^2)=16x-2x^3$. We also note that since we require the rectangle to be above the x-axis, we must have $0\le x\le\sqrt{8}$. Thus we can use the closed interval method to find the absolute minimum of A. $A'(x)=16-6x^2$, which is 0 when $16=6x^2$ or $x=\sqrt{\frac{8}{3}}$. $A(0)=A(\sqrt{8})=0$, and $A(\sqrt{\frac{8}{3}})\approx 17.4$, so the absolute maximum occurs at $x=\sqrt{\frac{8}{3}}$. At this point, $y=8-8/3=16/3$, so the rectangle giving the largest area is $2\sqrt{\frac{8}{3}}\times 16/3$.

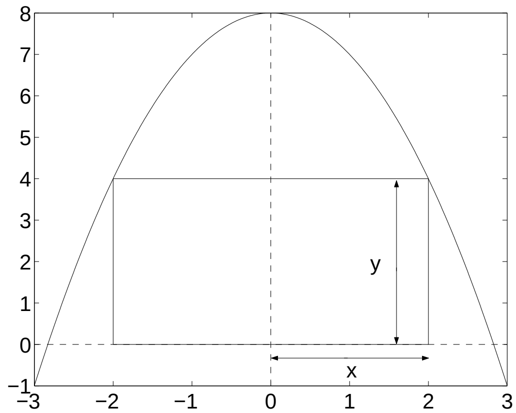


Figure 3: figure for #16

#32. We first note that we can find the fuel consumption per mile at a given speed by dividing the fuel consumption per hour by the speed: $G(v) = \frac{c}{v}$. In order to optimize G, we should take its first derivative and set it equal to 0, that is, $G'(v) = \frac{v\frac{dc}{dv} - c\frac{dv}{dv}}{v^2} = 0$. Multiplying through by v^2 and noting that $\frac{dv}{dv} = 1$, we have $v\frac{dc}{dv} - c = 0$, or $\frac{dc}{dv} = \frac{c}{v}$. We next note that the slope of a line passing through the origin and through any point (v, c) on the graph given in the book is $\frac{c}{v}$. Since we are looking for a point where $\frac{c}{v} = \frac{dc}{dv}$, we may equivalently look for a point where the line passing through (v, c) and the origin (which has slope $\frac{c}{v}$, recall) is one and the same as the tangent line to the curve $c = c(v)$ at the point (v, c). Doing this approximately and graphically gives us $v \approx 53$ mph.

Section 4.8

#4. **a**. If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, so Newton's method fails.

b. If $x_1 = 1$, then the tangent line is horizontal and Newton's method fails.

c. If $x_1 = 3$, then $x_2 = 1$, and we have the same situation as in (b). Newton's method fails again.

d. If $x_1 = 4$, then the tangent line is horizontal and Newton's method fails.

e. If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

#6. If $f(x) = x^3 - x^2 - 1$, then $f'(x) = 3x^2 - 2x$, so that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n^2 - 1}{3x_n^2 - 2x_n}.$$

Now, if $x_1 = 1$, then

$$x_2 = 1 - \frac{1 - 1 - 1}{3 - 2} = 2,$$

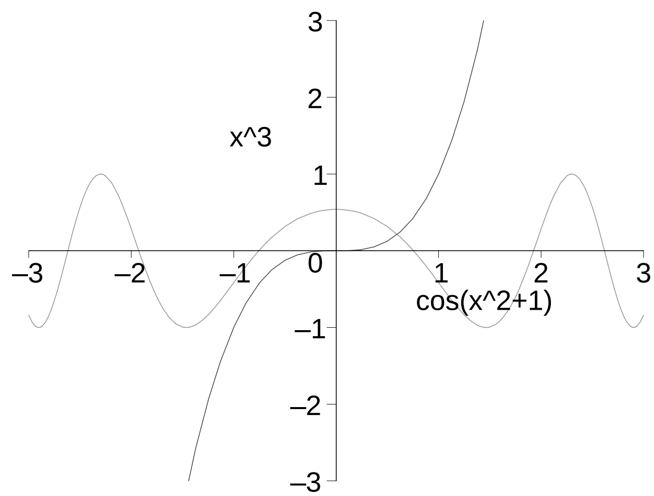
and

$$x_3 = 2 - \frac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 1.625.$$

#16. From the graph below, we see that the only root of this equation is near 0.6. Since $f(x) = \cos^3(x^2+1) - x^3$, we have $f'(x) = -2x\sin(x^2+1) - 3x^2$, so that

$$x_{n+1} = x_n - \frac{\cos^3(x_n^2+1) - x_n^3}{-2x_n \sin(x_n^2+1) - 3x_n^2}.$$

Taking $x_1 = 0.6$, we get $x_2 \approx 0.58688855$, $x_3 \approx 0.59698777 \approx x_4$. To eight decimal places, the root of the equation is 0.59698777.



#20. (a) If $f(x) = \frac{1}{x} - a$, then $f'(x) = -\frac{1}{x^2}$ so that

$$x_{n+1} = x_n - \frac{x_n^{-1} - a}{-x_n^{-2}} = x - n + x_n - xx_n^2 = 2x - n - ax_n^2.$$

(b) Using (a) with $a = 1000$ and $x_1 = 1/2 = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. Thus,

$$\frac{1}{1.6984} \approx 0.588789.$$

#24. If $f(x) = x^2 + \sin x$, then $f'(x) = 2x + \cos x$. $f'(x)$ exists for all x, so to find the minimum of f we can examine the zeroes of f'. From the graph of f', we see that a good choice for x_1 is $x - 1 = -0.5$. Use $g(x) = 2x + \cos x$ and $g'(x) = 2 - \sin x$ to obtain $x_2 \approx -0.450627$, $x_3 \approx -0.450184 \approx x_4$. Since $f''(x) = 2 - \sin x > 0$ for all x, $f(-0.450184) \approx -0.232466$ is the absolute minimum.

4. To show $\lim_{x\to 0} (\sec x)^{1/x^2} = \sqrt{e}$ we proceed as follows:

$$\begin{aligned}
\lim_{x\to 0} (\sec x)^{1/x^2} &= \lim_{x\to 0} e^{\ln(\sec x)^{1/x^2}} \\
&= e^{\lim_{x\to 0} \ln(\sec x)^{1/x^2}} \\
&= e^{\lim_{x\to 0} \frac{1}{x^2} \ln(\sec x)} \\
&= e^{\lim_{x\to 0} \frac{\ln(\sec x)}{x^2}}.
\end{aligned}$$

Now, we must compute $\lim_{x\to 0} \dfrac{\ln(\sec x)}{x^2}$ which is indeterminant $\left[\frac{0}{0}\right]$, so that we can use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{\ln(\sec x)}{x^2} = \lim_{x\to 0} \frac{\frac{\sec x \tan x}{\sec x}}{2x} = \lim_{x\to 0} \frac{\tan x}{2x} = \lim_{x\to 0} \frac{\sec^2 x}{2} = \frac{1}{2}.$$

Hence,

$$\lim_{x\to 0} (\sec x)^{1/x^2} = e^{1/2} = \sqrt{e}.$$

5. To compute $\lim_{x\to a} \left(\dfrac{\sin x}{\sin a}\right)^{1/(x-a)}$ which is indeterminant $[1^\infty]$, we proceed as above.

$$\begin{aligned}
\lim_{x\to a} \left(\frac{\sin x}{\sin a}\right)^{1/(x-a)} &= \lim_{x\to a} e^{\ln\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)}} \\
&= e^{\lim_{x\to a} \ln\left(\frac{\sin x}{\sin a}\right)^{1/(x-a)}} \\
&= e^{\lim_{x\to a} \frac{1}{(x-a)} \ln\left(\frac{\sin x}{\sin a}\right)} \\
&= e^{\lim_{x\to a} \frac{\ln\left(\frac{\sin x}{\sin a}\right)}{x-a}}
\end{aligned}$$

Now, we must compute $\lim_{x\to a} \dfrac{\ln\left(\frac{\sin x}{\sin a}\right)}{x-a}$ which is indeterminant $\left[\frac{0}{0}\right]$, so that we can use L'Hôpital's rule.

$$\lim_{x\to a} \frac{\ln\left(\frac{\sin x}{\sin a}\right)}{x-a} = \lim_{x\to a} \frac{\ln(\sin x) - \ln(\sin a)}{x-a} = \lim_{x\to a} \frac{\frac{\cos x}{\sin x}}{1} = \frac{\cos a}{\sin a}$$

since $\sin a \neq 0$.

Hence,

$$\lim_{x\to a} \left(\frac{\sin x}{\sin a}\right)^{1/(x-a)} = e^{\frac{\cos a}{\sin a}}.$$

6. **(a)** Let $f(x) = x^3 + 3x - 2k$. Then Newton's method tells us

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Since $f'(x) = 3x^2 + 3$, substituting yields

$$\begin{aligned}
x_{n+1} &= x_n - \frac{(x_n^3 + 3x_n - 2k)}{(3x_n^2 + 3)} \\
&= \frac{(3x_n^2 + 3)x_n - (x_n^3 + 3x_n - 2k)}{(3x_n^2 + 3)} \\
&= \frac{2}{3} \cdot \frac{x_n^3 + k}{x_n^2 + 1}
\end{aligned}$$

(b) Use the above formula with $k = 1$, $x_0 = 1$, to conclude

$$x_0 = 1$$
$$x_1 = \frac{2}{3} \cdot \frac{1^3 + 1}{1^2 + 1} = \frac{2}{3} \approx 0.66667$$
$$x_2 \approx 0.59829$$
$$x_3 \approx 0.59607$$
$$x_4 \approx 0.59607$$

$\therefore$ Accurate to 5 decimal places, $x^3 + 3x - 2 = 0$ has a root at 0.59607.

7. **(a)** Notice that $f(0)$ is not defined. However, $f(1) = -2 < 0$, $f(e) = e - 2 > 0$, and f is continuous on $[1, e]$. Thus, by the Intermediate Value Theorem, f has a root in $(1, e)$ [and therefore has a root in $(0, e)$].

(b) If $f(x) = x \ln x - 2$, then $f'(x) = \ln x + 1$. Hence, Newton's method tells us that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \ln x_n - 2}{\ln x_n + 1}.$$

Thus, $x_0 = 2$, $x_1 \approx 2.362464$, $x_2 \approx 2.345783$, $x_3 \approx 2.345751$, $x_4 \approx 2.345751$.

Accurate to six decimal places f has a root of 2.345751.

(c) Since $f'(x) = \ln x + 1$ and $f''(x) = 1/x$, if x is near 2 [in fact, if $x > 0$], then both $f'(x) > 0$ and $f''(x) > 0$ so that f is concave up and increasing. Thus, all tangent lines lie under the graph of f, and intersect the x-axis at points larger than the root. This implies that in the Newton's method scheme, all approximations will be bigger than the actual solution. [This would not be true if f were concave up and decreasing, instead.]