Math 111.01 Summer 2003
Assignment #4 Solutions

## 1. Practice problems.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.

## 2. Extra practice computing derivatives.

Solutions may be found in the back of the text, or in the *Student Solutions Manual*.

## 3. Problems to hand in.

**Section 3.5**

**#16.** $y' = e^{-5x}(-3\sin(3x)) - 5e^{-5x}\cos(3x)$

**#24.** $y' = \dfrac{(e^u + e^{-u})(2e^{2u}) - e^{2u}(e^u - e^{-u})}{(e^u + e^{-u})^2}$

**#38.** $w'(0) = (u \circ v)'(0) = u'(v(0))v'(0)$ by the chain rule.
Therefore $w'(0) = u'(2)(5) = (4)(5) = 20$.

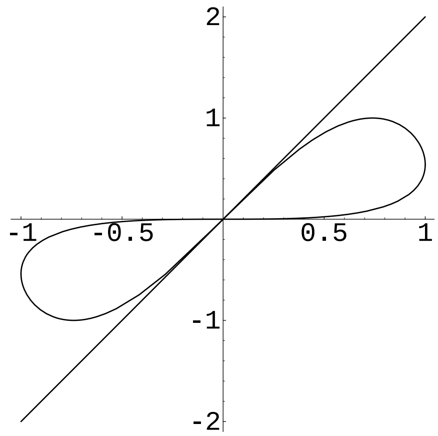
**#66.** To find the equation of the tangent lines at $(0,0)$ first we find the slope:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(\cos(t + \sin t))(1 + \cos t)}{\cos t}$$

To compute $\frac{dy}{dx}$ at the point $(0,0)$ we need to find the value of $t$ that gives $x = y = 0$. Since $x = \sin t$ then $x = 0$ when $t = 0, \pi$. Note that $y = 0$ also for $t = 0, \pi$. Therefore, evaluate $\frac{dy}{dx}$ when $t = 0, \pi$:

$$\begin{aligned} \frac{dy}{dx}(0) &= \frac{(1)(2)}{1} = 2 \\ \frac{dy}{dx}(\pi) &= \frac{0}{1} = 0 \end{aligned}$$

Since we have different slopes at $(0,0)$, we have 2 different tangent lines with equations $y = 2x$ and $y = 0$ as seen in the graph below (one of the tangent lines is the $x$-axis).




**Section 3.6**

**#6.**

$$\begin{aligned}
5y^4\frac{dy}{dx}+x^2(3y^2)\frac{dy}{dx}+2xy^3 &= y(2x)e^{x^2}+e^{x^2}\frac{dy}{dx}\\
\frac{dy}{dx}(5y^4+x^2(3y^2)-e^{x^2}) &= 2xye^{x^2}-2xy^3\\
\frac{dy}{dx} &= \frac{2xye^{x^2}-2xy^3}{5y^4+x^2(3y^2)-e^{x^2}}
\end{aligned}$$

**#14.**

$$\begin{aligned}
\frac{2x}{9}+\frac{2y}{36}\frac{dy}{dx} &= 0\\
\frac{y}{18}\frac{dy}{dx} &= \frac{-2x}{9}\\
\frac{dy}{dx} &= \frac{-4x}{y}
\end{aligned}$$

Evaluating $\frac{dy}{dx}$ at the point $(-1, 4\sqrt{2})$ gives $\frac{dy}{dx}=\frac{1}{\sqrt{2}}$. Therefore the equation of the tangent line is $y-4\sqrt{2}=\frac{1}{\sqrt{2}}(x+1)$.

**#26.** Taking the first derivative, $y'$ we get:

$$\begin{aligned}
2x+6xy'+6y+2yy' &= 0\\
y'(6x-2y) &= -2x-6y\\
y' &= \frac{-x-3y}{3x+y}
\end{aligned}$$

To find $y''$, use the quotient rule on $y'$:

$$\begin{aligned}
y'' &= \frac{(3x+y)(-1-3y')-(-x-3y)(3+y')}{(3x+y)^2}\\
&= \frac{(3x+y)(-1-3\frac{-x-3y}{3x+y})-(-x-3y)(3+\frac{-x-3y}{3x+y})}{(3x+y)^2}
\end{aligned}$$

**#32.** Using the fact that $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ (page 242) and the chain rule we have:

$$\begin{aligned} y' &= \frac{1}{1+(x-\sqrt{1+x^2})^2}\left(\frac{d}{dx}(x-\sqrt{1+x^2})\right) \\ &= \frac{1}{1+(x-\sqrt{1+x^2})^2}\left(1-\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)\right) \end{aligned}$$

**Section 3.7**

**#2.** $f'(x) = \frac{2x}{x^2+10}$

**#16.**

$$\begin{aligned} G(u) &= \ln\sqrt{\frac{3u+2}{3u-2}} = \frac{1}{2}\ln\left(\frac{3u+2}{3u-2}\right) = \frac{1}{2}\left(\ln(3u+2)-\ln(3u-2)\right) \\ G'(u) &= \frac{1}{2}\left(\frac{3}{3u+2}-\frac{3}{3u-2}\right) \end{aligned}$$

**#32.** $y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln x$. Now use implicit differentiation:

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x^2} + \ln x\left(-\frac{1}{x^2}\right) \\ \frac{dy}{dx} &= y\left(\frac{1-\ln x}{x^2}\right) \\ &= x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right) \end{aligned}$$

**#36.** $y = x^{\ln x} \Rightarrow \ln y = \ln x \ln x$. Now use implicit differentiation:

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x}\ln x + \frac{1}{x}\ln x \\ \frac{dy}{dx} &= y\frac{2\ln x}{x} \\ &= x^{\ln x}\left(\frac{2\ln x}{x}\right) \end{aligned}$$

**Section 2.9**

**#6.** **a**. Using the definition of derivative, we have:

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{(1+h)^3-h^3}{h} \\ &= \lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h} \\ &= \lim_{h\to 0}\frac{3h+3h^2+h^3}{h} \\ &= \lim_{h\to 0} 3+3h^1+h^2 = 3 \end{aligned}$$

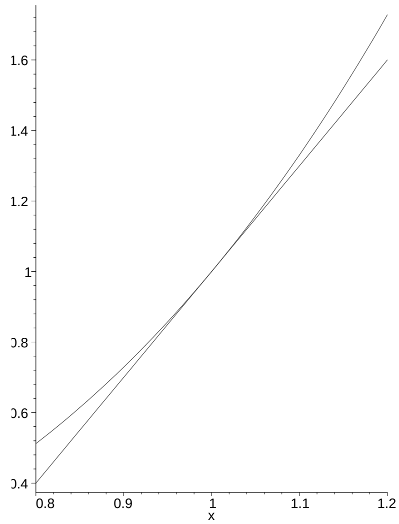
Of course, we also know some other differentiation rules now, so we know that the derivative of $x^3$ with respect to $x$ is $3x^2$. So $f'(1) = 3\cdot 1^2 = 3$.

**b**. The linear approximation is $L(x) = f(1) + f'(1)(x-1) = 1 + 3(x-1) = 3x - 2$.

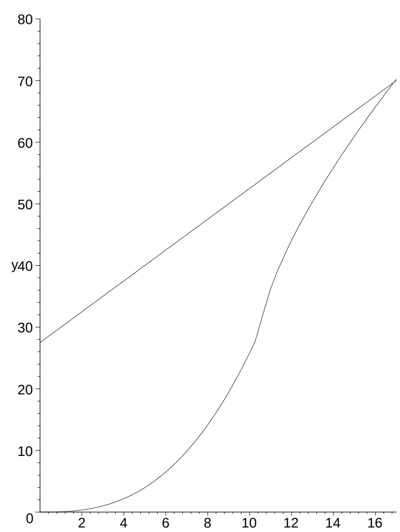
| $Linearization$ | $FunctionValue$ |
|---|---|
| $L(.9) = .7$ | $f(.9) = .729$ |
| $L(.95) = .85$ | $f(.95) = .957375$ |
| $L(.99) = .97$ | $f(.99) = .970299$ |
| $L(1.01) = 1.03$ | $f(1.01) = 1.030301$ |
| $L(1.05) = 1.15$ | $f(1.05) = 1.157625$ |
| $L(1.1) = 1.3$ | $f(1.1) = 1.331$ |

The linear approximation looks like it's a little bit too small.

**c**. The tangent line lies under this graph, so they really are underestimates!




**#12.** **a**. To get a linear approximation, we need to look at the graph and figure out what a good tangent line would be. To do this, we need to get a point and a slope for the tangent line. When the time is 17 weeks, there are about 70 thousand bees. We can approximate the slope of the tangent line by drawing one in and looking at it.



From this graph, the tangent line looks like to goes through the point (0,27). So the slope will be approximately $\frac{70-27}{17-0} \approx 2.5$ thousand bees per week. This yields a linearization of: $L(x) = f(17) + f'(17)(x-17) = 70 + 2.5(x-17)$. Thus the population at 18 weeks is approximately $L(18) = 70 + 2.5(18-17) = 72.5$ thousand bees. At 20 weeks the population is approximately $L(20) = 70 + 2.5(20-17) = 77.5$ thousand bees.

**b**. From the picture, we can see that the tangent line is above the graph, so these are overestimates.

**c**. Our estimate at 18 weeks should be better than the one at 20 weeks, since it's only one week after the data that we're given.

## Section 3.8

**#2.** $f(x) = \ln(x)$, so $f'(x) = \frac{1}{x}$. Hence, $f'(1) = 1$ and $f(1) = \ln(1) = 0$. Thus, our linearization is: $L(x) = f(1) + f'(1)(x-1) = 0 + 1(x-1) = x - 1$ .

**#16.** **a**. We use a linear approximation at $x = 1$. $f'(1) = \sqrt{1^3+1} = \sqrt{2}$. From this we find $L(x) = 2 + \sqrt{2}(x-1)$. So $f(1.1) \approx L(1.1) = 2 + \sqrt{2}(1.1-1) = 2 + \sqrt{2}(.1) \approx 2.1414$

**b**. Ordinarily, I'd like to look at the graph of $f(x)$ and compare it with the graph of $L(x)$, but here I cannot do that—I don't know what $f(x)$ looks like. I do know that when a function is concave up the tangent line is below the curve and when it's concave down the tangent line is above the curve. Whe can see which is the case by finding the second derivative: $f''(x) = 1/2(x^3+1)^{-1/2}3x^2 \geq 0$. Since the second derivative is positive, $f(x)$ is concave up, so the tangent line lies below the curve. Therefore, the actual value is greater than our approximation. Symbolically, $f(1.1) \geq 2.1414$

**#20.** **a**. The area of a circle is $A = \pi r^2$. We can find the relationship between the differentials by taking the derivative. $\frac{dA}{dr} = 2\pi r$ and so $dA = 2\pi r dr$. Since we want the maximum error in the area, we want to find $dA$. We know that the radius, $r$, is 24 cm, and the error in the radius, $dr$, is 0.2 cm. So $dA = 2\pi(24 \text{ cm})(.2 \text{ cm}) = 9.6\pi \text{ cm}^2 \approx 30 \text{ cm}^2$.

**b**. Relative error is $\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = 2\frac{dr}{r} = 2\frac{.2 \text{ cm}}{24 \text{ cm}} = \frac{1}{60} \approx 0.016667$. Percentage error is relative error $\times 100\% \approx 0.016667 \times 100\% = 1.6667\%$.

**#22.** Here $k$ is a constant, so we need to treat it as one when we take the derivative. $\frac{dF}{dR} = k4R^3$, and so $dF = k4R^3 dR$. Since $F = kR^4$, we find that the relative error is $\frac{\Delta F}{F} \approx \frac{dF}{F} = \frac{k4R^3 dR}{kR^4} = 4\frac{dR}{R} \approx 4\frac{\Delta R}{R}$, which is 4 times the relative error in the radius. So if we increase the radius by 5%, the flow will increase by about 4 times that amount, or 20 %.

## Section 4.1

**#4.** 1. What information is given?

"The $y$-coordinate is increasing at 4 m/s at the point $(2,3)$" tells us that

$$\frac{dy}{dt}(2,3) = 4$$

and we also know that the curve we are traveling along is $y = \sqrt{1+x^3}$.

2. What are we looking for?

"How fast is the $x$-coordinate changing at that point?" That is to say, what is $\frac{dx}{dt}(2,3)$?

So, we have $\frac{dy}{dt}$ and we want $\frac{dx}{dt}$. We can use the chain rule to derive a relationship between these derivatives as follows:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Next, we notice that we can find the missing piece of the above puzzle by finding $\frac{dy}{dx}$ from the equation for the curve along which the particle is moving.

$$\frac{dy}{dx} = \frac{1}{2}(1+x^3)^{-\frac{1}{2}}(3x^2)$$

and evaluating this at $x = 2$ for the point $(2,3)$ we find that $\frac{dy}{dx}(2,3) = 2$. Now we simply have to plug in what we know into our chain rule equation.

$$(4) = (2)\frac{dx}{dt}(2,3)$$

$$\frac{dx}{dt}(2,3) = 2$$

To interpret our solution, we say that we have now found that the $x$-coordinate of the particle is increasing at a rate of 2 m/s at the point $(2,3)$ in the curve.

**#6.** Begin by drawing a picture of the situation at noon on the coordinate plane so that the position of boat $A$is $(0,0)$ and the position of boat $B$ is $(150,0)$. With this picture we may make the following analysis:

What do we know?

If we let $x_A$ be the $x$-coordinate of ship $A$, then we know that $\frac{dx_A}{dt} = 35$ and the position of ship $A$ is $(x_A, 0)$, where $x_A$ is a function of $t$, since $A$ does not move up and down, only left and right. Next, we let $y_B$ be the $y$-coordinate of ship $B$, also a function of $t$, so that we know that $\frac{dy_B}{dt} = 25$. The position of ship $B$ is then $(150, y_B)$ since $B$ only moves up, not left and right. We will let $t = 0$ be 12 noon in what follows.

What do we want to know?

We want to know how fast the distance between the ships is changing at 4 p.m. So, let us compute this distance using the distance formula.

$$D(A,B) = \sqrt{(150 - x_A)^2 + (y_B - 0)^2}$$

Then what we want to find is: $\frac{dD(A,B)}{dt}$ at $t = 4$ hours.

What do we need to do to find this? Well, we can begin by using implicit differentiation to derive the equation for $D(A,B)$ with respect to $t$.

$$\frac{dD(A,B)}{dt} = \frac{1}{2}((150 - x_A)^2 + (y_B)^2)^{-\frac{1}{2}} \times (2(150 - x_A)(-\frac{dx_A}{dt}) + 2y_B\frac{dy_B}{dt}$$

Now we only need to find $x_a$, $y_B$, $\frac{dx_A}{dt}$, and $\frac{dy_B}{dt}$ at $t = 4$ hours. We can do this easily, since we know $\frac{dx_A}{dt}$, and $\frac{dy_B}{dt}$ at all times are the same. This tells us that ship $A$ will travel to the east $35 \times 4 = 140$ km by 4 p.m., so that at $t = 4$, $x_A = 140$, and ship $B$ will travel $25 \times 4 = 100$ km north by 4 p.m., so that at $t = 4$, $y_B = 100$. Now we simply substitute all of this information into the equation and solve for $\frac{dD(A,B)}{dt}$ to find that the solution is that the distance between the two ships is increasing at a rate of 42.79 km/h at 4 p.m.

**#8.** Begin with a drawing again. Draw a segment from 0 to 15 on the $y$-axis of a coordinate plane—this will represent our street lamp. Then choose a point somewhere on the positive part of the $x$-axis, let's call it $b$ and draw a line from the top of the street lamp to the point you choose—so draw from $(0, 15)$ to $(b, 0)$. This represents the light coming from the lamp. Then choose a point somewhere between 0 and $b$, and call this point $a$. Draw a vertical line up straight up from a to the segment we just drew. This vertical segment will represent the walking guy. Notice that the way we have set this up, that both the points $a$ and $b$ will be functions of time, since we will now start thinking of $b$ as the point of the tip of the shadow.

What do we know?

We know that the man is moving to the right, away from the pole at 5 ft/sec. This means that $\frac{da}{dt} = +5$. We also notice that we are staring at a picture of similar triangles. Therefore, we can set up the following relationship, using similar triangles for the two legs of the right triangles before us:

$$\frac{b}{15} = \frac{b-a}{6}$$

What do we want? We want to find the rate of change of the tip of the shadow, which is the point $b$. So, we are looking for $\frac{db}{dt}$ when $a = 40$. We will use implicit differentiation of the relationship we just found above, to get an equation which has $\frac{db}{dt}$ in it.

$$(1/15)\frac{db}{dt} = (1/6)(\frac{db}{dt} - \frac{da}{dt})$$

Now we find ourselves in great shape, since we ended up with an equation with $\frac{db}{dt}$ involving only $\frac{da}{dt}$, which is one of the quantities that we know. So, we can find $\frac{db}{dt}$ at any time we want, not just at the time when $a = 40$ ft, as asked in the problem. Our result is:

$$(1/15)\frac{db}{dt} = (1/6)(\frac{db}{dt} - (5))$$

$$\frac{db}{dt} = \frac{75}{9}$$

The tip of the shadow is moving away from the pole at a rate of 25/3 ft/sec.

**#14.** Begin with a drawing again. Let the dock be the $y$-axis, so draw a segment from 0 to 1 on the $y$-axis. Then place the boat at point $a$ along the positive part of the $x$-axis. Connect the points $(0, 1)$ and $(a, 0)$ with a line segment to represent the rope tied to the boat, and label this segment $R$ for rope.

What do we know?

We know that the boat is moving toward the dock in the water, so that the point $a$ will be a function of $t$ for us, and the distance from the dock to the boat will always be exactly $a$. We are given that the rope is being pulled in at a rate of 1 m/s, so we know that $\frac{dR}{dt} = -1$ m/s.

What do we want?

We want to find out how fast the boat is approaching the dock when the distance from the dock is 8 meters. So, we seek $\frac{da}{dt}$ when $a = 8$.

We see before us a right triangle, so we can use the Pythagorean Theorem to find that

$$R^2 = a^2 + 1^2.$$

If we differentiate this in time we obtain:

$$2R\frac{dR}{dt} = 2a\frac{da}{dt} + 0$$

We know $\frac{dR}{dt}$ and we want to use $a = 8$, so all we need to finish is to find R when $a = 8$, but we can use the equation above to find out that $R = \sqrt{65}$. Now we make our substitutions and use algebra to solve:

$$\frac{da}{dt} = -\sqrt{65}/8.$$

And we have found the rate at which the boat is approaching the dock when $a = 8$, in meters per second.

**#26.** What do we know? We have the equation $PV^{1.4} = C$.

What do we want? We want to know $\frac{dV}{dt}$ when $V = 400$, $P = 80$, and $\frac{dP}{dt} = -10$

Let's begin by using implict differentiation of our given relationship with respect to $t$. We need the product rule, and the chain rule.

$$\frac{dP}{dt}V^{1.4} + P(1.4)V^{0.4}\frac{dV}{dt} = 0$$

We can now simply plug in $V = 400$, $P = 80$, and $\frac{dP}{dt} = -10$ and solve to obtain

$$\frac{dV}{dt} == \frac{100}{7} = 14\frac{2}{7}.$$

**#28.** What do we know?

We are given two relationships:

$$B = .007W^{2/3}$$

$$W = .12L^{2.53}$$

Along with the information that the change in $L$ with respect to time has been constant over the last 10 million years. That means that the instantaneous rate of change is equal to the average rate of change, and so we can compute that $\frac{dL}{dt} = \frac{20-15}{10} = 0.5$ cm per million years.

What do we want?

We want to know $\dfrac{dB}{dt}$ when $L = 18$ cm.

We can complete this problem in two ways: we could substitute our expression for $W$ into the first equation to get $B$ in terms of $L$ alone—but this slightly complicates matters with the exponents. Instead, we will differentiate both equations with respect to time.

$$\frac{dB}{dt} = .007(2/3)W^{-1/3}\frac{dW}{dt}$$

$$\frac{dW}{dt} = .12(2.53)L^{1.53}\frac{dL}{dt}$$

We can use the second of our initial equations to find $W$ when $L = 18$ cm.

$$W = .12(18)^{2.53} = 179.9$$

Then we use the second of our derived equations to find $\frac{dW}{dt}$ when $L = 18$.

$$\frac{dW}{dt} = .12(2.53)(18)^{1.53}(.5) = 12.643,$$

and this answer comes in grams per million years. Finally, we can plug in all of this information to find our goal:

$$\frac{dB}{dt} = .007(2/3)(179.9)^{-1/3}(12.64) = .01045.$$

So, the brain size is changing at a rate of 5.23 grams per million years.

**Section 4.2**

**#2.** **a**. The Extreme Value Theorem states that $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$ if $f$ is continuous on $[a, b]$.

**b**. First find $f'(x)$. Next find the values of $x$ in the interval for which $f'(x)$ either is zero or does not exist. Then find the values of $f$ at these $x$-values, the critical points, as well as $f(a)$ and $f(b)$. The largest of these is the maximum, and the smallest is the minimum, as we have followed the steps of the closed interval method.

**#6.** By inspection we have local minima at $x = 1, 4, 6, -\frac{3}{4}$ with the global minimum at $x = 1$. Also by inspections local maxima occur at $x = 0, 3, 5, 7$ with the global maximum at $x = 7$.

**#14.** **a**. The graph of the function $f(x) = (x - 2)(x - 3)(x)(x + 2)$ with $x$-range from $-3$ to 4 is an example of such a function.

**b**. The graph of the function whose derivative is the following

$$((x - .5)^2)((x + .5)^2)(x)(x + 1)(x - 1)(x + 1.2)(x - 1.2)$$

Must have these properties, since it will be equal to 0 at exactly 7 places, and has two solutions of the seven which will not yield local extrema since the derivitave won't change sign at 0.5 or $-0.5$. Finally, the function will be a positive polynomial of degree 10, since this derivative will be positive of degree nine, and so it must have the required 3 minima and 2 maxima.

The upshot of all of this is, if we plot on our calculators the following function, from $x = -1.3$ to $x = 1.3$, we will find the picture that we want. This is an antiderivative of the expression above:

$$f(x) = .1x^{10} - .3675x^8 + .45375x^6 - .218125x^4 + .045x^2$$

**#24.**

$$f(x) = x^3 + x^2 - x$$
$$f'(x) = 3x^2 + 2x - 1$$

So, $f'(x) = 0$ when $x = 1/3$ or $x = -1$. The derivative of this function exists everywhere, so these are the only critical points.

**#34.**

$$f(x) = xe^{2x}$$
$$f'(x) = e^{2x} + xe^{2x}(2)$$

So, $f'(x) = 0$ when $x = -1/2$ and only there since $e^{2x}$ is never zero. This function also has derivative everywhere, and so there is only one critical point in this case.

**#48.**

$$f(x) = e^{x^3 - x}$$

And we observe the interval $-1 \le x \le 0$.

a) From the graph we observe a maximum on the interval at $y = 1.469$. The minimum is 1, atained at both 0 and -1

b)

$$f'(x) = (3x^2 - 1)e^{x^3 - x}$$

So, the critical point that is in the interval of interest is $x = -\sqrt{1/3}$. We need to evaluate only this point as well as the endpoints, 0 and 1 to find our maximum and minimum. We see that $f(-\sqrt{1/3}) = 1.469467\ldots$ and both of $f(0)$ and $f(-1)$ are 1. So, our maximum and minimum from part a are correct.

## Section 4.3

**#6.** **a**. $f$ is increasing when $f' > 0$. Looking at the graph, it is easy to see that $f$ is increasing on $(2, 4)$ and $(6, 9)$.

**b**. $f$ decreases until $x = 2$ and then increases, and thus there is a local minimum at $x = 2$. Similarly, there is a local minimum at $x = 6$. Note that $f$ increases from $x = 2$ to $x = 4$ and then decreases, so therefore there is a local maximum at $x = 4$.

**c**. $f$ is concave up when its second derivative is positive. Since $f''$ is the derivative of $f'$, $f''$ is positive exactly when $f'$ is increasing. Therefore we just need to look at the graph of $f'$ and see when it is increasing and decreasing. Hence $f$ is concave up on $(1, 3)\cup$ $(5, 7) \cup (8, 9)$, and $f$ is concave down on $(0, 1)\cup$ $(3, 5) \cup (7, 8)$.

**d**. The inflection points of $f$ are the points at which $f$ changes concavity. This occurs when $x$ is 1, 3, 5, 7, or 8.

**#14.** **a**. $f(x) = x \ln x$, so using the Product Rule, $f'(x) = x\frac{1}{x} + (\ln x)(1) = 1 + \ln x$. $f'(x) = 0$ when $\ln x = -1$, so $f'(x) = 0$ when $x = e^{-1}$. $f'(x) < 0$ for $x < e^{-1}$, and $f'(x) > 0$ for $x > e^{-1}$. Thus $f$ is decreasing on $(0, e^{-1})$ (note that the domain of $f$ is just $(0, \infty)$), and $f$ is increasing on $(e^{-1}, \infty)$.

**b**. $f$ decreases until $x = e^{-1}$ and then increases, so there is a local minimum at $(e^{-1}, f(e^{-1})) = (e^{-1}, -e^{-1})$.

**c**. $f''(x) = \frac{1}{x}$. The domain of $f$ is all positive real numbers, and thus $f''(x) > 0$ for all $x$ in the domain of $f$. Hence $f$ is always concave up, and there are no inflection points.

**#16.** **a**. Let $f(x) = x^4(x-1)^3$. Then

$$f'(x) = x^4(3(x-1)^2) + (x-1)^3(4x^3) = x^3(x-1)^2(3x + 4(x-1)) = x^3(x-1)^2(7x-4).$$

This expression is 0 when $x = 0, 1$, or $\frac{4}{7}$, so these are the critical numbers.

**b**. We need to compute $f''(x)$:

$$f''(x) = x^3(x-1)^2(7) + (7x-4)(x^3(2)(x-1) + (x-1)^2(3x^2)).$$

Now, $f''(0) = f''(1) = 0$, so the Second Derivative Test doesn't tell us anything about these two critical numbers. However, $f''(\frac{4}{7}) = (\frac{4}{7})^3(\frac{-3}{7})^2(7) > 0$, so $\frac{4}{7}$ is a local minimum.

**c**. The First Derivative Test gives more information. On $(-\infty, 0)$, $(\frac{4}{7}, 1)$, and $(1, \infty)$, $f'(x)$ is positive. On $(0, \frac{4}{7})$, $f'(x)$ is negative. So there's a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and $x = 1$ is neither a local maximum nor a local minimum.

**#26.** **a**. We have $f(x) = \frac{x}{(x-1)^2}$. Since

$$\lim_{x \to \pm\infty} \frac{x}{(x-1)^2} = 0,$$

$y = 0$ is a horizontal asymptote. Also,

$$\lim_{x \to 1} \frac{x}{(x-1)^2} = \infty,$$

$x = 1$ is a vertical asymptote.

**b**. We have

$$f'(x) = \frac{(x-1)^2 - x(2(x-1))}{(x-1)^4} = \frac{(x-1)^2 - 2x^2 + 2x}{(x-1)^4} = \frac{1-x^2}{(x-1)^4} = \frac{(1-x)(1+x)}{(x-1)^3} = \frac{-x-1}{(x-1)^3}.$$

Thus when $f'(x) = 0$, $-x - 1 = 0$, so $x = -1$. To find where $f$ is increasing or decreasing, we need to check the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Since $f'(x) < 0$ on $(-\infty, -1)$ and $(1, \infty)$, $f$ is decreasing on those intervals, and because $f'(x) > 0$ on $(-1, 1)$, $f$ is increasing on that interval.

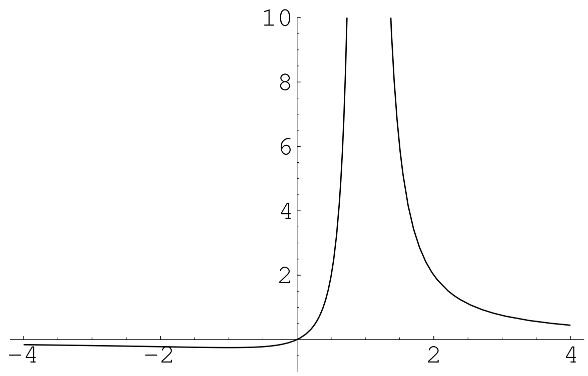
**c**. If $f'(x) = 0$, then $x = -1$. Note that $f$ is decreasing until $x = -1$ and increasing just after that, so $(-1, f(-1)) = (-1, \frac{-1}{4})$ is a local minimum.

**d**. We compute $f''(x)$:

$$f''(x) = \frac{(x-1)^3(-1) - (-x-1)(3(x-1)^2)}{(x-1)^6} = \frac{-(x-1) - 3(-x-1)}{(x-1)^4} = \frac{2x+4}{(x-1)^4}.$$

Setting this equal to 0, we find that $x = -2$. Note that $f''(x) < 0$ on the interval $(-\infty, -2)$, so $f$ is concave down there. Also, $f''(x) > 0$ on the intervals $(-2, 1)$ and $(1, \infty)$, and thus $f$ is concave up on those intervals. There is a concavity change when $x = -2$, so $(-2, f(-2)) = (-2, \frac{-2}{9})$ is an inflection point.

**e**. See the graph.




**#46.** By the Mean Value Theorem, there is a number $c$ in the interval $(2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}.$$

Since $1 \leq f'(x) \leq 4$ for all $x$ in $[2, 5]$,

$$1 \leq f'(c) \leq 4 \Longrightarrow 1 \leq \frac{f(5) - f(2)}{3} \leq 4 \Longrightarrow 3 \leq f(5) - f(2) \leq 12.$$

## Section 4.4

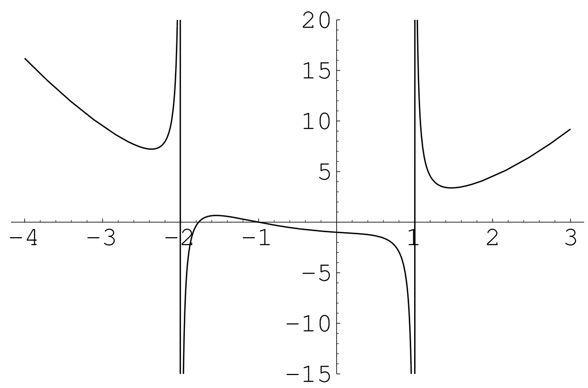
**#4.** We have

$$\begin{aligned} f'(x) &= \frac{(x^2+x-2)(4x^3+3x^2-4x) - (x^4+x^3-2x^2+2)(2x+1)}{(x^2+x-2)^2} \\ &= 2\frac{x^5+2x^4-3x^3-4x^2+2x-1}{(x^2+x-2)^2}, \end{aligned}$$

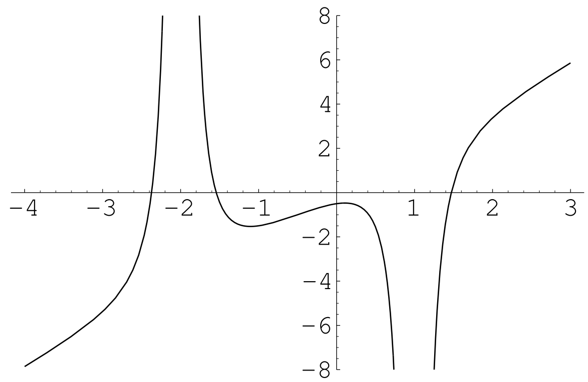
and also

$$\begin{aligned} &f''(x) \\ &= 2\frac{(x^2+x-2)^2(5x^4+8x^3-9x^2-8x+2) - (x^5+2x^4-3x^3-4x^2+2x-1)(2(x^2+x-2)(2x+1))}{(x^2+x-2)^4} \\ &= 2\frac{x^6+3x^5-3x^4-11x^3+12x^2+18x-2}{(x^2+x-2)^3}. \end{aligned}$$

The graph of $f$ looks like:

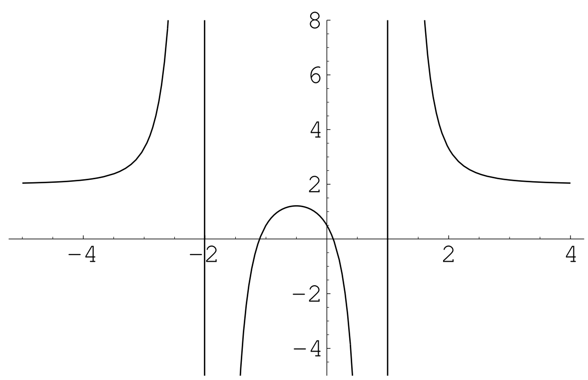



We consider the graph of $f'$ first:




Looking at where $f'$ is positive, we can estimate that $f$ is increasing on $(-2.4, -2)$, $(-2, -1.5)$, and $(1.5, \infty)$. It appears that $f'$ is negative approximately on the intervals $(-\infty, -2.4)$, $(-1.5, 1)$, and $(1, 1.5)$, and thus $f$ is decreasing on those intervals. Therefore there is one local maximum at approximately $(-1.5, 0.7)$. The local minima occur at around $(-2.4, 7.2)$ and $(1.5, 3.4)$.

To figure out the concavity of $f$, we use the graph of $f''$:

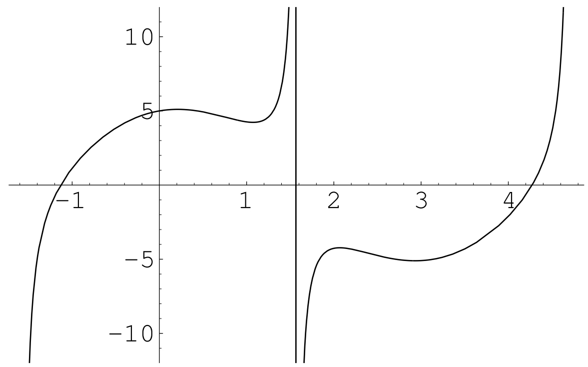



From the graph of $f''$, it appears that $f''$ is positive on the intervals $(-\infty, -2)$, $(-1.1, 0.1)$, and $(1, \infty)$, so $f$ is concave up there, and we estimate that $f$ is concave down on the intervals $(-2, -1.1)$ and $(0.1, 1)$, where $f''$ is negative. This means that $f$ has inflection points at around $(-1.1, 0.2)$ and $(0.1, -1.1)$.

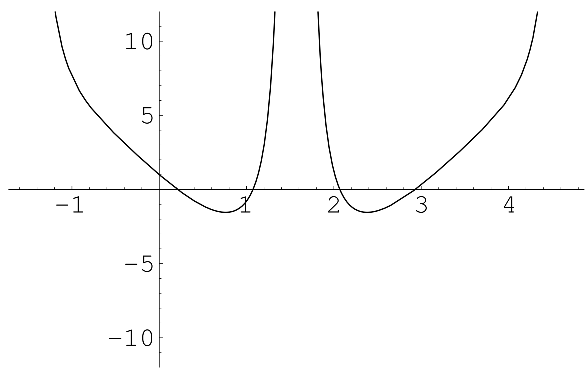
**#6.** Let $f(x) = \tan x + 5\cos x$. Then

$$f'(x) = \sec^2 x - 5\sin x, \text{ and } f''(x) = 2\sec x \sec x \tan x - 5\cos x = 2\sec^2 x \tan x - 5\cos x.$$

Note that $f$ has period $2\pi$, so we restrict our graphs to the intervals $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. The graph of $f$ looks like:

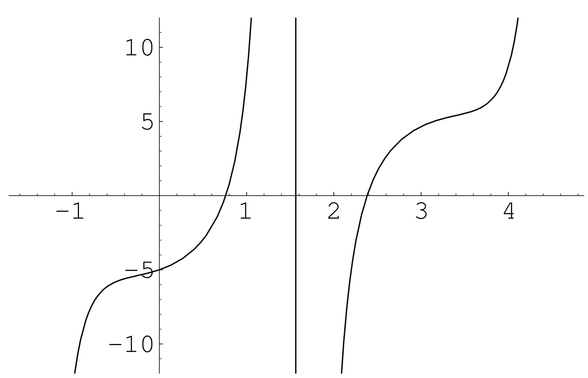



We use the graph of $f'$ to determine where $f$ is increasing and decreasing.



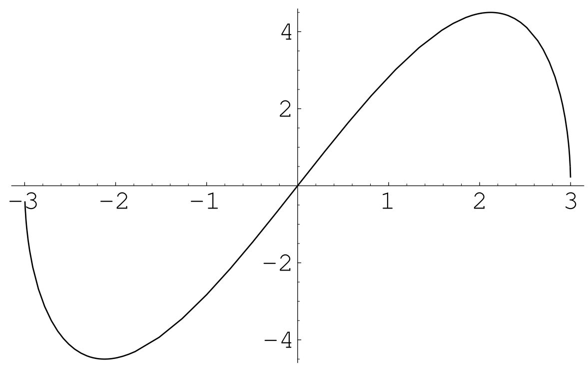

From the graph of $f'$, we estimate that $f$ is increasing on the intervals $(-\frac{\pi}{2}, 0.21)$, $(1.07, \frac{\pi}{2})$, $(\frac{\pi}{2}, 2.07)$, and $(2.93, \frac{3\pi}{2})$, and $f$ is decreasing on $(0.21, 1.07)$ and $(2.07, 2.93)$. This means that there are local minima at around $(1.07, 4.23)$ and $(2.93, -5.10)$, and there are local maxima at approximately $(0.21, 5.10)$ and $(2.07, -4.23)$.

To determine concavity, we look at where $f''$ is positive or negative.



We can estimate that $f$ is concave up on $(0.76, \frac{\pi}{2})$ and $(2.38, \frac{3\pi}{2})$, and $f$ is concave down on $(-\frac{\pi}{2}, 0.76)$ and $(\frac{\pi}{2}, 2.38)$. Thus $f$ has inflection points at about $(0.76, 4.57)$ and $(2.38, -4.57)$.

**#10.** We have $f(x) = x\sqrt{9-x^2}$. From the graph,




we can estimate that $f$ increases on about $(-2.1, 2.1)$ and decreases on $(-3, -2.1)$ and $(2.1, 3)$. (Note that the domain of $f$ is only $(-3, 3)$.) Hence $f$ has a local maximum at around $(2.1, 4.5)$ and a local minimum at around $(-2.1, -4.5)$. It also looks like $f$ is concave up on $(-3, 0)$ and concave down on $(0, 3)$, making $(0, 0)$ an inflection point.

Now we use calculus to determine these values exactly using the derivatives.

$$f'(x) = x(\frac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x)) + \sqrt{9-x^2} = \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2} = \frac{9-2x^2}{\sqrt{9-x^2}}.$$

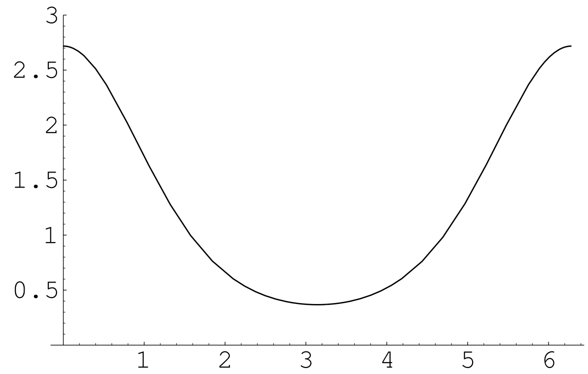
If $f'(x) = 0$, then $x = \pm\frac{3}{\sqrt{2}}$. The derivative is positive on the interval $(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, so $f$ is increasing there. $f$ is decreasing when the derivative is negative, on the intervals $(-3, -\frac{3}{\sqrt{2}})$ and $(\frac{3}{\sqrt{2}}, 3)$. This means that, using the First Derivative Test, $f$ has a local maximum at $(\frac{3}{\sqrt{2}}, \frac{9}{2})$ and a local minimum at $(-\frac{3}{\sqrt{2}}, -\frac{9}{2})$. Now,

$$f''(x) = \frac{\sqrt{9-x^2}(-2x) - (-x^2)(\frac{1}{2})(9-x^2)^{-\frac{1}{2}}(-2x)}{9-x^2} - x(9-x^2)^{-\frac{1}{2}} = \frac{-2x - x^3(9-x^2)^{-1} - x}{\sqrt{9-x^2}}$$

$$= \frac{-3x}{\sqrt{9-x^2}} - \frac{x^3}{(9-x^2)^{\frac{3}{2}}} = \frac{2x^3 - 27x}{(9-x^2)^{\frac{3}{2}}} = \frac{x(2x^2-27)}{(9-x^2)^{\frac{3}{2}}}.$$

This expression is positive on $(-3, 0)$, meaning $f$ is concave up there, and it is negative on $(0, 3)$, so $f$ is concave down on that interval, and there is an inflection point at $(0, 0)$.

**#12.** Since $f(x) = e^{\cos x}$ has period $2\pi$, we restrict our graphs to the interval $[0, 2\pi]$. Examining the graph of $f$,

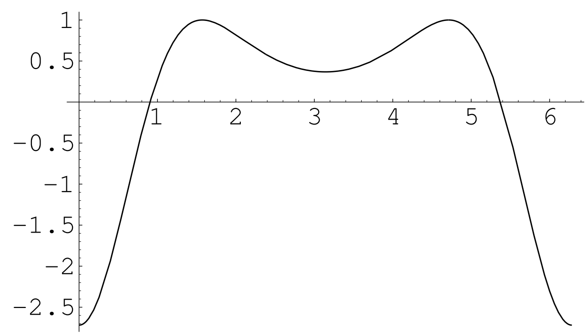



the local maxima seem to be at about $(0, 2.72)$ and $(2\pi, 2.72)$, and the local minimum is at around $(3.14, 0.37)$. We use the first derivative to find the precise values. $f'(x) = (-\sin x)e^{\cos x}$, which is 0 only when $-\sin x = 0$, and this occurs at 0, $\pi$, and $2\pi$ in the interval $[0, 2\pi]$. When $\pi < x < 2\pi$, $f'(x) > 0$, so $f$ is increasing on $(\pi, 2\pi)$. If $0 < x < \pi$, then $f'(x) < 0$, so $f$ is decreasing there. Therefore the local maxima on $[0, 2\pi]$ are $(0, e)$ and $(2\pi, e)$, and the local minimum on $[0, 2\pi]$ is at $(0, e^{-1})$.

To estimate the inflection points, we need to compute $f''(x)$.

$$f''(x) = (-\sin x)(e^{\cos x}(-\sin x)) + e^{\cos x}(-\cos x) = e^{\cos x}(\sin^2 x - \cos x).$$

From the graph




it appears that the inflection points occur when $x$ is around 0.90 and 5.38, giving inflection points at approximately $(0.90, 1.86)$ and $(5.38, 1.86)$.

**4.** We use the rules for differentiation to find

$$\begin{aligned}
f'(x) &= 2x|\cos\frac{\pi}{2x}| + x^2\frac{d}{dx}|\cos\frac{\pi}{2x}| \qquad \text{(product)} \\
&= 2x|\cos\frac{\pi}{2x}| + x^2\frac{|\cos\frac{\pi}{2x}|}{\cos\frac{\pi}{2x}}\frac{d}{dx}\cos\frac{\pi}{2x} \qquad \text{(chain) since } \frac{d}{dx}|x| = \frac{|x|}{x} \\
&= 2x|\cos\frac{\pi}{2x}| + x^2\frac{|\cos\frac{\pi}{2x}|}{\cos\frac{\pi}{2x}}(-\sin\frac{\pi}{2x})(-\frac{\pi}{2}\frac{1}{2x^2}) \\
&= 2x|\cos\frac{\pi}{2x}| + \frac{\pi}{4}\frac{|\cos\frac{\pi}{2x}|}{\cos\frac{\pi}{2x}}(-\sin\frac{\pi}{2x})
\end{aligned}$$

provided $x \neq 0$. Since $\frac{\pi}{2x}$ is well defined for $x \neq 0$, the only time $f'(x)$ is not well-defined is when there is division by 0; namely when $\cos\frac{\pi}{2x} = 0$. Now, $\cos\theta = 0$ when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, etc.

That is, $\cos\theta = 0$ when $\theta = \pm\frac{\pi}{2} \pm n\pi$, $n = 0, 1, 2, \ldots$.

Thus, $\cos\frac{\pi}{2x} = 0$ if $\frac{\pi}{2x} = \frac{\pi}{2} \pm n\pi$, or

$$\frac{1}{x} = 1 \pm 2n; \quad \text{or equivalently when,} \;\; x = \frac{1}{1 \pm 2n}.$$

Hence, $f'(x)$ is not defined if

$$x = 0, \pm 1, \pm\frac{1}{3}, \pm\frac{1}{5}, \pm\frac{1}{7}, \ldots = 0, 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, -1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, \ldots.$$

**5.** The linearization of a function $f(x)$ at the point $a$ is

$$L(x) = f'(a)(x-a) + f(a).$$

In this case,

$$\begin{aligned} L(x) &= f'(2)(x-2) + f(2) \\ &= -1(x-2) + 4 \\ &= 6 - x \end{aligned}$$

Therefore, the best guess we can make for $f(3)$ is

$$f(3) \approx L(3) = 6 - 3 = 3.$$

**6.** To complute $\frac{d}{dx}x^x$, let $y = x^x$. Then, $\ln y = \ln x^x = x\ln x$. Taking derivatives with respect to $x$ gives

$$\begin{aligned} &\frac{d}{dx}\ln y = \frac{d}{dx}(x\ln x) \\ &\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} \\ &\frac{dy}{dx} = y(\ln x + 1) \end{aligned}$$

Since $y = x^x$, we get

$$\frac{d}{dx}x^x = x^x(\ln x + 1).$$

**7.**

**(a)**

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x(1+3x^{-1})}{\sqrt{x^2}\sqrt{1+x^{-2}}} = \lim_{x\to\infty} \frac{1+3x^{-1}}{\sqrt{1+x^{-2}}} = 1.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x+3}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x(1+3x^{-1})}{\sqrt{x^2}\sqrt{1+x^{-2}}} = \lim_{x\to-\infty} -\frac{1+3x^{-1}}{\sqrt{1+x^{-2}}} = -1.$$

**(b)** Since

$$f'(x) = \frac{-3x+1}{\sqrt{(x^2+1)^3}}$$

we see that $f'(x) = 0$ when $x = 1/3$, and that for no $x$ is $f'$ undefined. Thus, the only critcal point is $x = 1/3$. Hence, if $x < 1/3$ then $f' > 0$, and if $x > 1/3$ then $f' < 0$.

Since

$$f''(x) = \frac{6x^2-3x-3}{\sqrt{(x^2+1)^5}}$$

we see that $f''(x) = 0$ when $2x^2 - x - 1 = (2x+1)(x-1) = 0$, or when $x = -1/2$ or $x = 1$. Hence, if $x < -1/2$ then $f'' > 0$, if $-1/2 < x < 1$ then $f'' < 0$, and if $x > 1$ then then $f'' > 0$.

Thus, there are no vertical asymptotes; there are horizontal asymptotes of $y = 1$ and $y = -1$; $x = 1/3$ is a local maximum by the Second Derivative Test; there are no minima; and there are inflection points at $x = 1$ and $x = -1/2$ since concavity changes at these points.