SECTION 10.3

8) *[10 pts]* First, we rearrange the transition matrix so that the absorbing states (states 1 and 3) come first:

$$\begin{array}{c} 1 \\ 3 \\ 2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline .6 & .3 & .1 \end{array}\right]$$

So now, we have that $R = [.6\ .3]$ and $Q = [.1]$. The fundamental matrix is then given by $F = (I_1 - Q)^{-1} = ([1] - [.1])^{-1} = [\frac{9}{10}]^{-1} = [\frac{10}{9}]$. So we have that $FR = [\frac{10}{9}][\frac{6}{10}\ \frac{3}{10}] = [\frac{2}{3}\ \frac{1}{3}]$.

16) Before anything else, we set up the transition matrix of the Markov Chain, and find its fundamental matrix. Keep in mind that state i is the state where person A has i dollars. Then the transition matrix is:

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The absorbing states are 0 and 5, so we rewrite the transition matrix accordingly:

$$\begin{array}{c} 0 \\ 5 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline \frac{3}{5} & 0 & 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \end{array}\right]$$

As before, we have

Also, we have:

$$FR = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \left[\begin{array}{cc} .9242 & .0758 \\ .8104 & .1896 \\ .6398 & .3602 \\ .3839 & .6161 \end{array}\right]$$

a) *[10 pts]* If B has 3 dollars then A has two dollars, and so the first entry in the second row of FR gives us the probability that A loses, which is .8104.

b) *[10 pts]* If B has 1 dollar, then A has four, and so the first entry in the last row of FR gives us the probability that A loses, which is .3839.

24a) *[5 pts]* We have that the transition matrix is:

$$\begin{matrix}1\\2\\3\end{matrix}\begin{bmatrix} .05 & .15 & .8 \\ .05 & .15 & .8 \\ 0 & 0 & 1 \end{bmatrix}$$

As usual, we rearrange the matrix so that the absorbing state (3) is first:

$$\begin{matrix}3\\1\\2\end{matrix}\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline .8 & .05 & .15 \\ .8 & .05 & .15 \end{array}\right]$$

So, we have:

$$R = \begin{bmatrix} .8 \\ .8 \end{bmatrix}; Q = \begin{bmatrix} .05 & .15 \\ .05 & .15 \end{bmatrix}$$

b) *[10 pts]* By definition, we have:

c) *[5 pts]* The probability that the disease eventually disappears is 1, by the definition of an absorbing Markov chain! (Also, you can look in the matrix FR to see that the probability that state 2 ends up in state 3 is 1.)

d) *[5 pts]* The expected number of people infected before absorption into state 3 is the sum of the entries in row 2 of FR, $.0625 + 1.1875 = 1.25$.

CHAPTER 10 PRACTICE PROBLEMS

12) *[15 pts total; 5 for the 'two repetitions,' and 10 for the 'long range trend']*

$$D = \begin{bmatrix} .8 & .2 \end{bmatrix}; T = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix}\begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} = \begin{bmatrix} .55 & .45 \\ .3 & .7 \end{bmatrix}$$

So now,

$$DT^2 = \begin{bmatrix} .8 & .2 \end{bmatrix}\begin{bmatrix} .55 & .45 \\ .3 & .7 \end{bmatrix} = \begin{bmatrix} .5 & .5 \end{bmatrix}$$

And this gives us the distribution after two repetitions. To find the long-range distribution, we need to find the equilibrium vector $V = [x\ y]$. (We know such a vector exists since this Markov chain is regular.) Now V must satisfy:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

But this is equivalent to the system of equations:

Also, since the equilibrium vector is a probability vector, we have $x + y = 1$, and so $x = 1 - y$. Plugging this into either of the above equations gives $y = \frac{3}{5}$, meaning $x = \frac{2}{5}$, and so we now have the long-range distribution: $V = [\frac{2}{5}\ \frac{3}{5}]$.

24) *[5 pts]* It is clear that the transition matrix below has no absorbing states (i.e., there are no 1s on the diagonal). So, the associated Markov chain is not absorbing.

$$\begin{bmatrix} .5 & .1 & .1 & .3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ .1 & .8 & .05 & .05 \end{bmatrix}$$

32) *[10 pts]*

$$X_0 = \begin{bmatrix} .4 & .4 & .2 \end{bmatrix}; A = \begin{bmatrix} .8 & .15 & .05 \\ .25 & .55 & .2 \\ .04 & .21 & .75 \end{bmatrix}; A^2 \begin{bmatrix} .6795 & .213 & .1075 \\ .3455 & .382 & .2725 \\ .1145 & .279 & .6065 \end{bmatrix}$$

To find the distribution after two months (i.e., two iterations of the chain), just multiply the initial probability vector by the square of the transition matrix. This gives:

$$X_0 A^2 = \begin{bmatrix} .4 & .4 & .2 \end{bmatrix} \begin{bmatrix} .6795 & .213 & .1075 \\ .3455 & .382 & .2725 \\ .1145 & .279 & .6065 \end{bmatrix} = \begin{bmatrix} .4329 & .2398 & .2733 \end{bmatrix}$$

34) *[15 pts]* To find the long-range distribution (i.e., equilibrium vector), use the same procedure outlined in exercise 12. This gives a system of equations:

Combining like terms in each equation and then solving this system by the Gauss-Jordan method gives $x = \frac{47}{114}$, $y = \frac{32}{114}$, and $z = \frac{35}{114}$. So, the long-range trend is $[\frac{47}{114}\ \frac{32}{114}\ \frac{35}{114}]$.