An Introduction to the Loewner Equation and SLE

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Charles Loewner

- 1893: born May 29 as Karl Löwner in Lany, Bohemia
- 1917: Ph.D. from University of Prague in geometric function theory under Georg Pick
- 1933: jailed during Nazi occupation of Prague, emmigrated to US, changed his name to Charles Loewner, and received Assistant Professorship at Louisville University
- Brown University (1944-1946); Syracuse University (1946-1951); Stanford University (1951-1968)
Brief History of the Loewner Equation

• (Loewner 1923): proved a special case of the Bieberbach conjecture ($|a_3| \leq 3$)

• (DeBranges 1985): proved entire Bieberbach conjecture

• (Schramm 1999): scaling limits of certain stochastic processes

• (Lawler, Schramm, Werner 2000): proved Mandelbrot’s conjecture that dimension of Brownian frontier is $4/3$
Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane.

Let $h : \mathbb{H} \to \mathbb{H}$ be onto with $h(\infty) = \infty$.

Then $h$ must be of the form $h(z) = az + b$ where $a > 0$ and $b \in \mathbb{R}$. 

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**Mapping $\mathbb{H}$ to $\mathbb{H}$**

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane.
Riemann Mapping Theorem

The Riemann mapping theorem states that any simply connected proper subset of the complex plane can be mapped conformally onto the unit disk, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).

**Theorem** (Riemann). Let \( D \) be a simply connected domain which is a proper subset of complex plane. Let \( z_0 \in D \) be a given point. Then there exists a unique analytic function \( g \) which maps \( D \) conformally onto \( \mathbb{D} \) and has the properties \( g(z_0) = 0 \) and \( g'(z_0) > 0 \).
Mapping $\mathbb{H} \setminus K$ to $\mathbb{H}$

Suppose $K$ is a bounded, compact set such that $\mathbb{H} \setminus K$ is simply connected.

By the Riemann mapping theorem there exist many conformal maps $g_K$ from $\mathbb{H} \setminus K$ to $\mathbb{H}$ with $g_K(\infty) = \infty$.

Using the Schwarz reflection principle, as $z \to \infty$ we can expand $g_K$ around $\infty$.

$$
\therefore g_K(z) = bz + a_0 + \frac{a_1}{z} + O\left(\frac{1}{z^2}\right)
$$

with $b > 0$ and $a_i \in \mathbb{R}$.

Consider the expansion of $f(z) = [g_K(1/z)]^{-1}$ about the origin. $f$ locally maps $\mathbb{R}$ to $\mathbb{R}$ so the coefficients in the expansion are real and $b > 0$. 
For convenience, we choose the unique $g_K$ which satisfies the “hydrodynamic normalization”

$$\lim_{z \to \infty} (g_K(z) - z) = 0.$$ 

i.e. we choose $b = 1$, $a_0 = 0$

The constant $a(K) := a_1$ only depends on the set $K$.

Thus $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ with $g_K(\infty) = \infty$ is

$$g_K(z) = z + \frac{a(K)}{z} + O\left(\frac{1}{z^2}\right).$$
Let \( \gamma : [0, \infty) \to \mathbb{H} \) be a simple curve (no self intersections) with \( \gamma(0) = 0 \) and \( \gamma(0, \infty) \subseteq \mathbb{H} \).

For each \( t \geq 0 \) suppose that \( K_t := \gamma[0, t] \).

Let \( \mathbb{H}_t := \mathbb{H} \setminus K_t \) be the slit half plane and let \( g_t : \mathbb{H}_t \to \mathbb{H} \) be the corresponding Riemann map.

We want \( g_t(\infty) = \infty \) and \( g_t \) to satisfy hydrodynamic normalization.

Thus as \( z \to \infty \),

\[
g_t(z) = z + \frac{a(t)}{z} + O \left( \frac{1}{z^2} \right)
\]

where \( a(t) := a(K_t) \).
The slit half plane $\mathbb{H}_t$ and the corresponding Riemann map to $\mathbb{H}$.

- The curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- $K_t := \gamma[0, t]$, $\mathbb{H}_t := \mathbb{H} \setminus K_t$, $g_t : \mathbb{H}_t \to \mathbb{H}$
- $U_t := g_t(\gamma(t))$, the image of $\gamma(t)$.
- By the Carathéodory extension theorem, $g_t(\gamma[0, t]) \subseteq \mathbb{R}$. 
Understanding $a(t)$

As before $K_t = \gamma[0, t]$, $H_t = H \setminus K_t$, and

$$a(t) := a(K_t) = a(\gamma[0, t]).$$

We choose the parametrization of $\gamma(t)$ such that $a(t) = 2t$.

i.e. Let $\sigma_t = \inf\{s : a(\gamma(s)) = 2t\}$. Then $\sigma_t$ is such that $a(\gamma[0, \sigma_t]) = 2t$.

Reparametrize by $\tilde{\gamma}(t) = \gamma(\sigma_t)$. Just call this $\gamma$.

**Facts.**

1) if $s < t$, then $a(s) < a(t)$

2) $s \mapsto a(s)$ is continuous

3) $a(0) = 0$, $a(t) \to \infty$ as $t \to \infty$
The Loewner Equation

Assume that $\gamma(t)$ is chosen so that $a(t) = 2t$.

Suppose $K_t := \gamma[0, t]$ with $\mathbb{H}_t := \mathbb{H} \setminus K_t$ and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding maps. Let $U_t := g_t(\gamma(t))$.

Then $g_t$ satisfies the Loewner differential equation with the identity map as initial data.

**Theorem** (Loewner). $g_t(z)$, for fixed $z$, is the solution of the IVP

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$
The natural thing to do is to start with $U_t$ and solve the Loewner equation.

Suppose that the function $t \mapsto U_t$, $t \in [0, \infty)$ is continuous and real-valued.

Solving the Loewner equation gives $g_t$ which conformally map $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$. Although for many choices of $U$ this is not possible, the following theorem gives a sufficient condition.

**Theorem** (Rohde-Marshall). *If $U$ is “nice” [Hölder 1/2 continuous with sufficiently small Hölder 1/2 norm], then $\gamma(t) = g_t^{-1}(U_t)$ is a well-defined simple curve and $K_t = \gamma[0, t]$.  

Brownian motion is a model of “continuous, random motion.” Think of Brownian motion as the limit of a random walk where the step sizes get smaller and smaller (and the grid gets finer and finer).

Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ for all $i$.

If $S_n = X_1 + X_2 + \cdots + X_n$, then $\frac{S_n}{\sqrt{n}} \to \text{BM}$ (in distribution as $n \to \infty$).

One dimensional Brownian motion is a real-valued process on the line; $B : [0, \infty) \to \mathbb{R}$, $B_0 = 0$, $B(t) = B_t$. 
Facts about Brownian motion.

1) $t \mapsto B_t$ is continuous

2) $B_t \sim \mathcal{N}(0, t)$, $\sigma B_t \sim B_{\sigma^2 t} \sim \mathcal{N}(0, \sigma^2 t)$

3) $B_{t+s} - B_s \sim \mathcal{N}(0, t)$ (stationary increments)

4) $B_t$ is independent of $B_s$ for $0 \leq s < t$ (independent increments)

5) $-B_t$ is a Brownian motion

6) $B_t$ is Hölder $\alpha$ continuous for all $0 < \alpha < 1/2$
SLE

• Stochastic Loewner Evolution (aka Schramm’s LE) introduced by Oded Schramm in 1999
• developed by Lawler, Schramm, Werner and Rohde, Marshall

The idea: let $U_t$ be a Brownian motion!

SLE with parameter $\kappa$ is obtained by choosing $U_t = \sqrt{\kappa}B_t$ where $B_t$ is a standard one dimensional Brownian motion.

**Definition.** SLE$_{\kappa}$ in the upper half plane is the random collection of conformal maps $g_t$ obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \quad g_0(z) = z.$$
It is not obvious that $g_t^{-1}$ is well-defined at $U_t$ so that the curve $\gamma$ can be defined. The following theorem establishes this.

**Theorem** (Rohde-Schramm). *There exists a curve $\gamma$ associated to SLE$_\kappa$ (at least for $\kappa \neq 8$).*

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa}B_t)$.

SLE$_\kappa$ is the random collection of conformal maps $g_t$ (complex analysts) or the curve $\gamma[0,t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter $\kappa$ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.
Properties of SLE

Fact.

- $0 < \kappa \leq 4$: $\gamma(t)$ can be defined and is a random, simple curve.
- $4 < \kappa < 8$: $\gamma(t)$ can be defined, but it is not a simple curve. It has double points, but does not cross itself!
- $\kappa > 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Conjecture. The Hausdorff dimension of the paths $\gamma(t)$ depends on $\kappa$.

- $\dim_H(SLE_\kappa) = 1 + \frac{\kappa}{8}$ for $\kappa < 8$
- $\dim_H(SLE_\kappa) = 2$ for $\kappa > 8$

Of course $\dim_H(SLE_\kappa) = 2$ for $\kappa > 8$ since it is space-filling.
References


