Intersection probabilities for a chordal SLE path and a semicircle

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This talk is based on joint work with Tom Alberts of the Courant Institute.
Review of SLE

Let $\mathbb{H} = \{ z \in \mathbb{C} : \Im(z) > 0 \}$ denote the upper half plane, and consider a simple (non-self-intersecting) curve $\gamma : [0, \infty) \to \mathbb{H}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$.

For every fixed $t \geq 0$, the slit plane $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0, t]$ is simply connected and so by the Riemann mapping theorem, there exists a unique conformal transformation $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$ which can be expanded as

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty,$$

where $b(t) = \text{hcap}(\gamma(0, t])$ is the half-plane capacity of $\gamma$ up to time $t$.

It can be shown that there is a unique point $U_t \in \mathbb{R}$ for all $t \geq 0$ with $U_t := g_t(\gamma(t))$ and that the function $t \mapsto U_t$ is continuous.
Review of SLE (cont)

\[ g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty, \quad \mathbb{H}_t = \mathbb{H} \setminus \gamma(0,t) \]

The evolution of the curve \( \gamma(t) \), or more precisely, the evolution of the conformal transformations \( g_t : \mathbb{H}_t \to \mathbb{H} \), can be described by a PDE involving \( U_t \).

This is due to C. Loewner (1923) who showed that if \( \gamma \) is a curve as above such that its half-plane capacity \( b(t) \) is \( C^1 \) and \( b(t) \to \infty \) as \( t \to \infty \), then for \( z \in \mathbb{H} \) with \( z \notin \gamma[0,\infty) \), the conformal transformations \( \{ g_t(z), t \geq 0 \} \) satisfy the PDE

\[ \frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \]

Note that if \( b(t) \in C^1 \) is an increasing function, then we can reparametrize the curve \( \gamma \) so that \( hcap(\gamma(0,t]) = b(t) \). This is the so-called parametrization by capacity.
\[
\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. 
\] (\star)

The obvious thing to do now is to start with a continuous function \( t \mapsto U_t \) from \([0, \infty)\) to \( \mathbb{R} \) and solve the Loewner equation for \( g_t \).

Ideally, we would like to solve (\star) for \( g_t \), define simple curves \( \gamma(t), \ t \geq 0 \), by setting \( \gamma(t) = g_t^{-1}(U_t) \), and have \( g_t \) map \( \mathbb{H} \setminus \gamma(0, t] \) conformally onto \( \mathbb{H} \).

Although this is the intuition, it is not quite precise because we see from the denominator on the right-side of (\star) that problems can occur if \( g_t(z) - U_t = 0 \).

Formally, if we let \( T_z \) be the supremum of all \( t \) such that the solution to (\star) is well-defined up to time \( t \) with \( g_t(z) \in \mathbb{H} \), and we define \( \mathbb{H}_t = \{ z : T_z > t \} \), then \( g_t \) is the unique conformal transformation of \( \mathbb{H}_t \) onto \( \mathbb{H} \) with \( g_t(z) - z \to 0 \) as \( t \to \infty \).
The novel idea of Schramm was to take the continuous function $U_t$ to be a one-dimensional Brownian motion starting at 0 with variance parameter $\kappa \geq 0$.

The *chordal Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard parametrization* (or simply $\text{SLE}_\kappa$) is the random collection of conformal maps $\{g_t, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} W_t}, \quad g_0(z) = z,$$

where $W_t$ is a standard one-dimensional Brownian motion.
Review of SLE (cont)

The question is now whether there exists a curve associated with the maps $g_t$.

- If $0 < \kappa \leq 4$, then there exists a random simple curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$, i.e., the curve $\gamma(t) = g_t^{-1}(\sqrt{\kappa B_t})$ never re-visits $\mathbb{R}$. As well, the maps $g_t$ obtained by solving $(\ast)$ are conformal transformations of $\mathbb{H} \setminus \gamma(0, t]$ onto $\mathbb{H}$. For this range of $\kappa$, our intuition matches the theory!

- For $4 < \kappa < 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$. These curves have double points and they do hit $\mathbb{R}$, but they never cross themselves! As such, $\mathbb{H} \setminus \gamma(0, t]$ is not simply connected. However, $\mathbb{H} \setminus \gamma(0, t]$ does have a unique connected component containing $\infty$. This is $\mathbb{H}_t$ and the maps $g_t$ are conformal transformations of $\mathbb{H}_t$ onto $\mathbb{H}$. We think of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ where $K_t$ is the hull of $\gamma(0, t]$ visualized by taking $\gamma(0, t]$ and filling in the holes.

- For $\kappa \geq 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ which is space-filling! Furthermore, it has double points, but does not cross itself! As in the case $4 < \kappa < 8$, the maps $g_t$ are conformal transformations of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ where $K_t$ is the hull of $\gamma(0, t]$.

As a result, we also refer to the curve $\gamma$ as chordal SLE$_\kappa$. SLE paths are extremely rough: the Hausdorff dimension of a chordal SLE$_\kappa$ path is $\min\{1 + \kappa/8, 2\}$. 
Review of SLE (cont)

Since there exists a curve $\gamma$ associated with the maps $g_t$, it is possible to reparametrize it.

It can be shown that if $U_t$ is a standard one-dimensional Brownian motion, then the solution to the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2/\kappa}{g_t(z) - U_t} = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

is chordal $\text{SLE}_\kappa$ parametrized so that $\text{hcap}(\gamma(0, t]) = 2t/\kappa = at$.

Finally, chordal SLE as we have defined it can also be thought of as a measure on paths in the upper half plane $\mathbb{H}$ connecting the boundary points $0$ and $\infty$.

SLE is conformally invariant and so we can define chordal $\text{SLE}_\kappa$ in any simply connected domain $D$ connecting distinct boundary points $z$ and $w$ to be the image of chordal $\text{SLE}_\kappa$ in $\mathbb{H}$ from $0$ to $\infty$ under a conformal transformation from $\mathbb{H}$ onto $D$ sending $0 \mapsto z$ and $\infty \mapsto w$. 
\[ \kappa = 1 \]

\[ \kappa = 2 \]
\[ \kappa = \frac{8}{3} \]

\[ \kappa = 3 \]
MK was interested in multiple SLEs and wanted to estimate the diameter of a chordal SLE path in $\mathbb{H}$ connecting the boundary points 0 and $x > 0$.

TA was interested in Hausdorff dimension and wanted to estimate the probability that a chordal SLE path in $\mathbb{H}$ connecting 0 and $\infty$ intersected a semicircle centred on the real line.

The two problems are the same.

Ideally, we hoped to determine these results asymptotically ($\sim$), but could only get them up to constants ($\asymp$).

Note. $\sim$ implies $\asymp$ implies $\approx$. 
The main estimate

**Theorem.** Let \( x > 0 \) be real, \( 0 < r \leq 1/3 \), and \( C(x; rx) = \{x + rxe^{i\theta} : 0 < \theta < \pi\} \) denote the semicircle of radius \( rx \) centred at \( x \) in the upper half plane, and suppose that \( \gamma : [0, \infty) \to \mathbb{H} \) is a chordal \( \text{SLE}_\kappa \) in \( \mathbb{H} \) from 0 to \( \infty \).

(a) If \( 0 < \kappa < 8 \), then \( P\{\gamma[0, \infty) \cap C(x; rx) \neq \emptyset\} \asymp r^{\frac{8-\kappa}{\kappa}} \).

(b) If \( \kappa = 8/3 \), then \( P\{\gamma[0, \infty) \cap C(x; rx) \neq \emptyset\} = 1 - (1 - r^2)^{5/8} \sim \frac{5}{8} r^2 \).
Corollary. Let $x > 0$ be real, $R \geq 3$, and $C(0; Rx) = \{ Rx e^{i\theta} : 0 < \theta < \pi \}$ denote the circle of radius $Rx$ centred at $0$ in the upper half plane, and suppose that $\gamma' : [0, 1] \to \mathbb{H}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from $0$ to $x$.

(a) If $0 < \kappa < 8$, then $P\{ \gamma'[0, 1] \cap C(0; Rx) \neq \emptyset \} \asymp R^{\kappa - 8/\kappa}$.

(b) If $\kappa = 8/3$, then $P\{ \gamma'[0, 1] \cap C(0; Rx) \neq \emptyset \} = 1 - (1 - R^{-2})^{5/8} \sim \frac{5}{8} R^{-2}$. 
Derivation of the corollary

The idea is to determine the appropriate sequence of conformal transformations and use the conformal invariance of chordal SLE.

Suppose that $\gamma' : [0, 1] \to \overline{\mathbb{H}}$ is an SLE$_\kappa$ in $\mathbb{H}$ from 0 to $x > 0$. Note that we are not interested in the parametrization of the SLE path, but only in the points visited by its trace. Suppose that $R \geq 3$, and consider $C(0; Rx) = \{Rx e^{i\theta} : 0 < \theta < \pi\}$. For $z \in \mathbb{H}$, let

$$ h(z) = \frac{R^2}{R^2 - 1} \frac{z}{x - z} $$

so that $h : \mathbb{H} \to \mathbb{H}$ is a conformal (Möbius) transformation with $h(0) = 0$ and $h(x) = \infty$. It is straightforward (though tedious) to verify that

$$ h(C(0; Rx)) = C(-1; \frac{1}{R}). $$

\[\text{Diagram}\]
Derivation of the corollary (cont.)

If \( \gamma : [0, \infty) \to \overline{\mathbb{H}} \) is a chordal \( \operatorname{SLE}_\kappa \) in \( \mathbb{H} \) from 0 to \( \infty \), then the conformal invariance of \( \operatorname{SLE} \) implies that

\[
P\{\gamma'[0, 1] \cap C(0; Rx) \neq \emptyset\} = P\{h(\gamma'[0, 1]) \cap h(C(0; Rx)) \neq \emptyset\}
= P\left\{\gamma[0, \infty) \cap C\left(-1, \frac{1}{R}\right) \neq \emptyset\right\}.
\]

By the symmetry of \( \operatorname{SLE} \) about the imaginary axis,

\[
P\left\{\gamma[0, \infty) \cap C\left(-1, \frac{1}{R}\right) \neq \emptyset\right\} = P\left\{\gamma[0, \infty) \cap C\left(1, \frac{1}{R}\right) \neq \emptyset\right\} \asymp R^{1-4\alpha}.
\]
The key fact that is needed is the restriction property of chordal SLE_{8/3}.

**Fact. [Lawler-Schramm-Werner]** If $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE_{8/3} in $\mathbb{H}$ from 0 to $\infty$, and $A$ is a bounded subset of $\mathbb{H}$ such that $\mathbb{H} \setminus A$ is simply connected, $A = \mathbb{H} \cap \overline{A}$, and $0 \notin \overline{A}$, then

$$P\{\gamma[0, \infty) \cap A = \emptyset\} = \left[\Phi'_A(0)\right]^{5/8}$$

where $\Phi_A : \mathbb{H} \setminus A \to \mathbb{H}$ is the unique conformal transformation of $\mathbb{H} \setminus A$ to $\mathbb{H}$ with $\Phi_A(0) = 0$ and $\Phi_A(z) \sim z$ as $z \to \infty$. 

![Diagram](image)
This implies that

\[ P\{\gamma[0, \infty) \cap C(x; rx) = \emptyset\} = \left[\Phi'(0)\right]^{5/8} \]

where \( \Phi = \Phi_{D(x; rx)}(z) \) is the conformal transformation from \( \mathbb{H} \setminus D(x; rx) \) onto \( \mathbb{H} \) with \( \Phi(0) = 0 \) and \( \Phi(z) \sim z \) as \( z \to \infty \).

\[ \mathbb{H} \setminus D(x; rx) \quad \Phi : \mathbb{H} \setminus D(r; rx) \to \mathbb{H} \]

In fact, the exact form of \( \Phi(z) \) is given by

\[ \Phi(z) = z + \frac{r^2 x^2}{z - x} + r^2 x. \]

Note that \( \Phi(0) = 0 \), \( \Phi(\infty) = \infty \), and \( \Phi'(\infty) = 1 \). We calculate \( \Phi'(0) = 1 - r^2 \) and therefore conclude that

\[ P\{\gamma[0, \infty) \cap C(x; rx) = \emptyset\} = (1 - r^2)^{5/8}. \]
**Rephrasing the main estimate**

**Theorem.** Let $x > 0$ be a fixed real number, and suppose $0 < \epsilon \leq x/3$. If $\gamma : [0, \infty) \to \mathbb{H}$ is a chordal SLE$_{\kappa}$ in $\mathbb{H}$ from 0 to $\infty$ with $0 < \kappa < 8$ and $a = 2/\kappa$, then

$$P\{\gamma[0, \infty) \cap C(x; \epsilon) \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}$$

where $C(x; \epsilon)$ is the semicircle of radius $\epsilon$ centred at $x$ in the upper half plane.

Written in this form, it is seen to generalize the result of Rohde and Schramm who prove that for $4 < \kappa < 8$,

$$P\{\gamma[0, \infty) \cap [x - \epsilon, x + \epsilon] \neq \emptyset\} \asymp \left(\frac{\epsilon}{x}\right)^{4a-1}.$$
An application

Let $0 < r \leq 1/3$, and suppose that $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to $\infty$ with $4 < \kappa < 8$ and $a = 2/\kappa$.

**Theorem.** There exist constants $c'_a$ and $c''_a$ such that

$$1 - c'_a r^{4a-1} \leq \inf_{z \in C_r} P\{T_z = T_1\} \leq \sup_{z \in C_r} P\{T_z = T_1\} \leq 1 - c''_a r^{4a-1}$$

where

$$C_r = C \left(1 - r; \frac{r}{2}\right)$$

denotes the circle of radius $r/2$ centred at $1 - r$ in the upper half plane.

**Corollary.** There exist constants $c'_a$ and $c''_a$ such that

$$1 - c'_a r^{4a-1} \leq P\{T_z = T_1 \text{ for all } z \in C_r\} \leq 1 - c''_a r^{4a-1}.$$
Proof of the application

The proof follows by combining the main result with a method due to Dubédat.

Suppose that $0 < r \leq 1/3$ and consider the two semicircles

$$C_r = C \left(1 - r; \frac{r}{2}\right)$$

and

$$C'_r = C \left(1 - \frac{3r}{4}; \frac{3r}{4}\right).$$
Proof of the application (lower bound)

It follows from the rephrased main result that

\[ P\{\gamma[0, \infty) \cap C'_r \neq \emptyset\} \asymp r^{4a-1} \]

and so there exists a constant \( c'_a \) such that

\[ 1 - c'_a r^{4a-1} \leq P\{\gamma[0, \infty) \cap C'_r = \emptyset\}. \]

However, it clearly follows that

\[ P\{\gamma[0, \infty) \cap C'_r = \emptyset\} \leq \inf_{z \in C'_r} P\{T_z = T_1\} \]

where \( T_z \) is the swallowing time of the point \( z \in \overline{H} \) (and the infimum is over all \( z \in C_r \text{ not } z \in C'_r \)). From this we conclude that there exists a constant \( c'_a \) such that

\[ 1 - c'_a r^{4a-1} \leq \inf_{z \in C_r} P\{T_z = T_1\}. \]
Proof of the application (upper bound)

In order to derive an upper bound, we use a method due to Dubédat.

Let $g_t$ denote the solution to the chordal Loewner equation with driving function $U_t = -B_t$ where $B_t$ is a standard one-dimensional Brownian motion with $B_0 = 0$. For $t < T_1$, the swallowing time of the point 1, consider the conformal transformation $\tilde{g}_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ given by

$$\tilde{g}_t(z) = \frac{g_t(z) + B_t}{g_t(1) + B_t}, \quad \tilde{g}_0(z) = z.$$ 

Note that $\tilde{g}_t(\gamma(t)) = 0$, $\tilde{g}_t(1) = 1$, $\tilde{g}_t(\infty) = \infty$, and that $\tilde{g}_t(z)$ satisfies the stochastic differential equation

$$d\tilde{g}_t(z) = \left[ \frac{a}{\tilde{g}_t(z)} + (1 - a)\tilde{g}_t(z) - 1 \right] \frac{dt}{(g_t(1) + B_t)^2} + [1 - \tilde{g}_t(z)] \frac{dB_t}{g_t(1) + B_t}.$$ 

If we now perform a time-change and also denoted the time-changed flow by $\{\tilde{g}_t(z), \ t \geq 0\}$, then then $\tilde{g}_t(z)$ satisfies the SDE

$$d\tilde{g}_t(z) = \left[ \frac{a}{\tilde{g}_t(z)} + (1 - a)\tilde{g}_t(z) - 1 \right] dt + [1 - \tilde{g}_t(z)] dB_t$$

Dubédat showed that for all $\kappa > 0$, this does not explode in finite time (wp1).
Therefore, if $F$ is an analytic function on $\mathbb{H}$ such that $\{F(\tilde{g}_t(z)), \ t \geq 0\}$ is a local martingale, then Itô’s formula implies that $F$ must be a solution to the differential equation

$$w(1 - w)F''(w) + [2a - (2 - 2a)w]F'(w) = 0.$$ 

An explicit solution is given by

$$F(w) = \frac{\Gamma(2a)}{\Gamma(1 - 2a)\Gamma(4a - 1)} \int_0^w \zeta^{-2a}(1 - \zeta)^{4a - 2} d\zeta$$

which is normalized so that $F(0) = 0$ and $F(1) = 1$.

Note that this is a Schwarz-Christoffel transformation of the upper half plane onto the isosceles triangle whose interior angles are $(1 - 2a)\pi, (1 - 2a)\pi,$ and $(4a - 1)\pi$. 


Proof of the application (upper bound) (cont.)

If

\[ F(w) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^w \zeta^{-2a}(1-\zeta)^{4a-2} \, d\zeta, \]

then the vertices of the triangle are at \( F(0) = 0 \), \( F(1) = 1 \), and

\[ F(\infty) = \frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(2-4a)\Gamma(4a-1)} e^{(1-2a)\pi i}. \]
Apply the optional sampling theorem to the martingale $F(\tilde{g}_{t \wedge T_z \wedge T_1}(z))$ to find that for $z \in \mathbb{H}$,

$$F(\tilde{g}_0(z)) = F(z) = F(0)P\{T_z < T_1\} + F(1)P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\}$$

$$= P\{T_z = T_1\} + F(\infty)P\{T_z > T_1\}. \quad (*)$$

Consequently, identifying the imaginary and real parts of (*) implies that

$$\Re\{F(z)\} = P\{T_z = T_1\} + \Re\{F(\infty)\}P\{T_z > T_1\}.$$

Since $\Re\{F(\infty)\} \geq 0$, we conclude $P\{T_z = T_1\} \leq \Re\{F(z)\} \leq |F(z)|$.

But now integrating along the straight line from $0$ to $z$ gives

$$|F(z)| \leq 1 - \frac{\Gamma(2a)}{\Gamma(1 - 2a)\Gamma(4a - 1)} \int_{|z|}^{1} \rho^{-2a}(1 - \rho)^{4a-2}d\rho$$

which relied on the fact that $4a - 2 < 0$. 

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Proof of the application (upper bound) (cont.)

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Proof of the application (upper bound) (cont.)

If \( z \in C_r \) so that \( 0 < 1 - \frac{3r}{2} \leq |z| \leq 1 - \frac{r}{2} < 1 \) by definition, then

\[
\int_{|z|}^{1} \rho^{-2a} (1 - \rho)^{4a-2} d\rho \geq \frac{2^{1-4a}}{4a - 1} r^{4a-1}.
\]

Hence,

\[
P\{T_z = T_1\} \leq |F(z)| \leq 1 - c''_{a} r^{4a-1}
\]

where

\[
c''_{a} = \frac{2^{1-4a}}{4a - 1} \frac{\Gamma(2a)}{\Gamma(1 - 2a) \Gamma(4a - 1)}.
\]

Taking the supremum of the previous expression over all \( z \in C_r \) gives us the required upper bound.