1 Variation of Brownian motion

Let $f : [a, b] \to \mathbb{R}$ be a real-valued function defined on the interval $a \le t \le b$, and suppose that $\Delta_n := \{a = t_0 < t_1 < \cdots t_{n-1} < t_n = b\}$ is a partition of $[a, b]$. Define the *mesh* of the partition Δ_n by

$$||\Delta_n|| := \max_{1 \le i \le n} (t_i - t_{i-1}).$$

For every $p > 0$, let

$$Q_p(f; a, b, \Delta_n) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p .$$

Our goal is to investigate the limiting behaviour of $Q_p(B; a, b, \Delta_n)$ where B is a Brownian motion. There are a number of different notions of "limit" that need to be considered, however.

Definition 1.1. The *true pth variation of f on $[a, b]$* is defined as

$$V_p(f; a, b) := \sup_{\Delta_n} Q_p(f; a, b, \Delta_n)$$

where the supremum is over all possible partitions of $[a, b]$.

If $V_p(f; a, b) < \infty$, then we say that f has *finite true pth variation on* $[a, b]$. In the particular case when $p = 1$, if $V_1(f; a, b) < \infty$, we say that f is of *bounded variation on* $[a, b]$ or has *finite total variation on* $[a, b]$.

It is known (see Taylor, Duke Math. J., 1972) that for Brownian motion

$$V_p(B; a, b) < \infty \quad \Leftrightarrow \quad p > 2.$$

The case $p = 2$ is of particular interest. Although the true 2nd variation of the Brownian path is unbounded, if instead of considering the supremum over all possible partitions, we restrict ourselves to those sequences of partitions $\{\Delta_n\}$ for which $||\Delta_n|| \to 0$ then something quite different is found.

Theorem 1.2. *If $\{\Delta_n,\ n = 1, 2, 3, \ldots\}$ is a sequence of partitions of $[a, b]$, then*

$$Q_2(B; a, b, \Delta_n) \to b - a \quad \text{in } L^2$$

as $||\Delta_n|| \to 0$.

Proof. To begin, notice that

$$\sum_{i=1}^{n} (t_i - t_{i-1}) = b - a.$$

Let

$$Y_n = \sum_{i=1}^{n} \left|B_{t_i} - B_{t_{i-1}}\right|^2 - (b - a) = \sum_{i=1}^{n} \left[\left|B_{t_i} - B_{t_{i-1}}\right|^2 - (t_i - t_{i-1})\right] = \sum_{i=1}^{n} X_i$$

where

$$X_i = \left|B_{t_i} - B_{t_{i-1}}\right|^2 - (t_i - t_{i-1}),$$

and note that

$$Y_n^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i\, X_j = \sum_{i=1}^{n} X_i^2 + 2 \sum_{i<j} X_i\, X_j.$$

The independence of the Brownian increments implies that $\mathbb{E}(X_i X_j) = 0$ for $i \neq j$; hence,

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2).$$

But

$$\begin{aligned}
\mathbb{E}(X_i^2) &= \mathbb{E}\left(B_{t_i} - B_{t_{i-1}}\right)^4 - 2(t_i - t_{i-1})\mathbb{E}\left(B_{t_i} - B_{t_{i-1}}\right)^2 + (t_i - t_{i-1})^2 \\
&= 3(t_i - t_{i-1})^2 - 2(t_i - t_{i-1})^2 + (t_i - t_{i-1})^2 \\
&= 2(t_i - t_{i-1})^2
\end{aligned}$$

since the fourth moment of a normal random variable with mean 0 and variance $t_i - t_{i-1}$ is $3(t_i - t_{i-1})^2$. Therefore,

$$\mathbb{E}(Y_n^2) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) = 2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \leq 2\, ||\Delta_n||\, \sum_{i=1}^{n} (t_i - t_{i-1}) = 2(b-a)\, ||\Delta_n|| \to 0$$

as $||\Delta_n|| \to 0$ from which we conclude that $Y_n \to 0$ in L^2; that is, $Q_2(B; a, b, \Delta_n) \to b - a$ in L^2 as $||\Delta_n|| \to 0$. □

As a result of this theorem, we define the quadratic variation of Brownian motion to be this L^2-limit.

Definition 1.3. The *quadratic variation* of a Brownian motion B on the interval $[a, b]$ is defined to be

$$Q_2(B; a, b) := \lim_{||\Delta_n|| \to 0} Q_2(B; a, b, \Delta_n) \quad \text{in } L^2.$$

Corollary 1.4. *If* $\{\Delta_n,\ n = 1, 2, 3, \ldots\}$ *is a sequence of partitions of* $[a, b]$ *with*

$$\sum_{n=1}^{\infty} ||\Delta_n|| < \infty,$$

then

$$Q_2(B; a, b, \Delta_n) \to b - a \quad a.s.$$

Proof. Suppose that $\epsilon > 0$. It follows from Chebychev's inequality that

$$\sum_{n=1}^{\infty} P(|Y_n| > \epsilon) \leq \frac{1}{\epsilon} \sum_{n=1}^{\infty} \leq \frac{2(b-a)}{\epsilon} \sum_{n=1}^{\infty} ||\Delta_n|| < \infty$$

using the notation in the proof of the previous theorem. By the Borel-Cantelli Lemma, we therefore conclude that $Y_n \to 0$ a.s. □

Remark. For example, if the interval $[0,1]$ is partitioned by the dyadic rationals

$$\Delta_n = \left\{ \frac{k}{2^n},\ k = 0, \ldots, 2^n \right\},$$

then

$$\sum_{n=1}^{\infty} ||\Delta_n|| = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

so that $Q_2(B; 0, 1, \Delta_n) \to 1$ a.s.

Corollary 1.5. *If $\{\Delta_n,\ n = 1, 2, 3, \ldots\}$ is a sequence of partitions of $[a,b]$, then*

$$Q_1(B; a, b, \Delta_n) = \sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right| \to \infty \quad a.s.$$

as $||\Delta_n|| \to 0$. In other words, almost all Brownian paths are of unbounded variation on every time interval.

Proof. Suppose to the contrary that B is a function of bounded variation, and let $V_1(B; a, b)$ denote the total variation of B on the interval $[a,b]$. It then follows that

$$\sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right|^2 \le \max_{1 \le i \le n} |B_{t_i} - B_{t_{i-1}}| \sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right| \le V(a,b) \max_{1 \le i \le n} |B_{t_i} - B_{t_{i-1}}|.$$

Since B is continuous a.s. on $[a,b]$, it is necessarily uniformly continuous on $[a,b]$. Therefore,

$$\max_{1 \le i \le n} |B_{t_i} - B_{t_{i-1}}| \to 0 \quad \text{as } ||\Delta_n|| \to 0$$

from which we conclude that

$$\sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right|^2 \to 0 \ \text{ a.s.}$$

This is a contradiction to the previous corollary, and establishes the result. $\square$