University of Regina
Department of Mathematics & Statistics
Final Examination
200710

Statistics 862
Stochastic Processes

Instructor: Michael Kozdron

This exam is due at 5:00 pm on Friday, April 20, 2007, in my office (College West 307.31).

Read all of the following information before starting this exam.

Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit.

You may use standard notation; however, any new notations or abbreviations that you introduce must be clearly defined. You may cite results from our textbook Introduction to Stochastic Processes*, second edition, by G. Lawler, without recopying them.*

You may consult any notes or books or websites that you wish, provided that proper acknowledgements are included. However, you may not discuss this exam with anyone before 5:00 pm on April 20, 2007. This includes students, professors, and colleagues.

*This exam has **6** problems, and it is worth a total of **100** points.*

1. (*20 points*) Consider a discrete time Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ which moves according to the following rules.

- If the chain is currently in state 0, then it chooses one of the five states 1, 2, 3, 4, 5 randomly, i.e., each with probability $1/5$.
- If the chain is currently in state $k > 0$, then it chooses one of the k states $0, 1, 2, \ldots, k-1$ randomly, i.e., each with probability $1/k$.

(a) Write down the transition matrix for this Markov chain. Is it irreducible? Is it aperiodic?

(b) What is the invariant probability for the chain?

(c) Suppose that the chain starts in state 0. What is the expected number of steps until the chain reaches 0 again?

(d) Suppose that the chain starts in state 0. What is the expected number of steps until the chain reaches either state 3 or state 4?

(e) Suppose that the chain starts in state 0. What is the probability that the chain will be in state 3 before its first visit to state 4?

2. (*20 points*) Consider branching processes with the following offspring distributions:

$$\text{I.}\quad p(0) = 1/4, \quad p(2) = 1/2, \quad p(4) = 1/4;$$

$$\text{II.}\quad p(n) = \frac{1}{4}\left(\frac{3}{4}\right)^n, \quad n = 0, 1, 2, \ldots;$$

$$\text{III.}\quad p(n) = \frac{e^{-1}}{n!}, \quad n = 0, 1, 2, \ldots;$$

$$\text{IV.}\quad p(0) = 3/4, \quad p(3) = 1/4.$$

(a) In each case, find the probability of eventual extinction assuming that the population starts with one individual.

(b) In case III, find the probability that the population survives at least 200 generations given that it survives at least 100 generations.

3. (*15 points*) Consider a continuous time Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition rates

$$\alpha(n, n+2) = 1, \quad n = 1, 2, \ldots,$$

$$\alpha(0, 2) = 5,$$

$$\alpha(n, n-1) = 4, \quad n = 1, 2, \ldots.$$

(a) Is this chain positive recurrent? If so, find the invariant probability $\overline{\pi}$.

(b) Is this chain transient? If so, find the probability starting at 5 of ever returning to the origin.

(c) Suppose that the chain starts in state 0. What is the probability that by time 5 the chain has been in exactly two states, 0 and 2?

4. (*10 points*) Consider the following simple game. You roll a single die. If the die comes up "1" you lose. If the die comes up any other number k, then you can either take a payoff of k^2 dollars or you can play again. Hence, the final payoff of the game is either 0 (if you ever roll a 1) or the square of the final roll when you quit.

(a) What is the optimal strategy in this game?

(b) Suppose that it costs r dollars to play this game each time. What is the smallest value of r such that the optimal strategy is to play when one rolls a 2 and to stop after any other roll?

5. (*20 points*) Consider the following sequence $M_0, M_1, M_2, \ldots$ of random variables. Suppose that $p, q \in (0, 1)$ and set $M_0 = p$. Suppose further that the distribution of M_{n+1} depends only on M_n by

$$P(M_{n+1} = (1-q)M_n \mid M_n) = 1 - M_n,$$
$$P(M_{n+1} = q + (1-q)M_n \mid M_n) = M_n.$$

(a) Show that $\{M_n,\ n = 0, 1, \ldots\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$, the information contained in $M_0, \ldots, M_n$.

(b) Show that $0 \le M_n < 1$ for each $n = 0, 1, \ldots$. Use this fact to explain why the limit

$$\lim_{n\to\infty} M_n = M_\infty$$

exists (in distribution).

(c) Determine the distribution of M_∞.

6. (*15 points*) Text, Exercise 8.4 on pages 195–196