Stat 800 Spring 2006
Solutions to Assignment #3

Solution to Exercise (1.5.6). Suppose that T is geometric with killing rate $1-\lambda$ so that $P\{T=j\} = (1-\lambda)\lambda^j$ as in Section 1.3. Therefore, by definition of $G_\lambda(x,y)$

$$G_\lambda(x,y) = \sum_{k=0}^{\infty} P^x\{S_k = y, T \geq k\} = \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} P^x\{S_k = y, T = j\} = \sum_{j=0}^{\infty}\sum_{k=0}^{j} P^x\{S_k = y, T = j\}$$

Since T is assumed to be independent of S, it follows that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{j} P^x\{S_k = y, T = j\} = \sum_{j=0}^{\infty}\sum_{k=0}^{j} P^x\{S_k = y\}P\{T=j\} = \sum_{j=0}^{\infty} P\{T=j\}\sum_{k=0}^{j} P^x\{S_k = y\}$$
$$= \sum_{j=0}^{\infty} P\{T=j\}G_j(x,y) = \sum_{j=0}^{\infty}(1-\lambda)\lambda^j G_j(x,y)$$

as required.

An alternative proof can be given as follows. Using the definition of $G_j(x,y)$, we find

$$\sum_{j=0}^{\infty}(1-\lambda)\lambda^j G_j(x,y) = \sum_{j=0}^{\infty}(1-\lambda)\lambda^j \sum_{k=0}^{j} P^x\{S_k = y\} = \sum_{j=0}^{\infty}\sum_{k=0}^{j}(1-\lambda)\lambda^j P^x\{S_k = y\}$$
$$= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}(1-\lambda)\lambda^j P^x\{S_k = y\} = (1-\lambda)\sum_{k=0}^{\infty} P^x\{S_k = y\}\sum_{j=k}^{\infty}\lambda^j.$$

But

$$\sum_{j=k}^{\infty}\lambda^j = \sum_{j=0}^{\infty}\lambda^j - \sum_{j=0}^{k-1}\lambda^j = \frac{1}{1-\lambda} - \frac{1-\lambda^k}{1-\lambda} = \frac{\lambda^k}{1-\lambda}$$

so that

$$(1-\lambda)\sum_{k=0}^{\infty} P^x\{S_k = y\}\sum_{j=k}^{\infty}\lambda^j = (1-\lambda)\sum_{k=0}^{\infty}\frac{\lambda^k}{1-\lambda}P^x\{S_k = y\} = \sum_{k=0}^{\infty}\lambda^k P^x\{S_k = y\} = G_\lambda(x,y)$$

as required.

Solution to Exercise (1.5.7). Using the definition of σ_x we find

$$G_A(x,x) = E^x[\sum_{j=0}^{\infty} I\{S_j = x, \tau > j\}] = 1 + E^x[\sum_{j=\sigma_x}^{\infty} I\{S_j = x, \tau > j\}].$$

However, conditioning on the value of σ_x gives

$$E^x[\sum_{j=\sigma_x}^{\infty} I\{S_j = x, \tau > j\}] = \sum_{k=1}^{\infty} E^x[\sum_{j=\sigma_x}^{\infty} I\{S_j = x, \tau > j\} \,|\, \sigma_x = k, \tau > k]\; P^x\{\sigma_x = k, \tau > k\}$$

By the strong Markov property (Theorem 1.3.2),

$$E^x[\sum_{j=\sigma_x}^{\infty} I\{S_j = x, \tau > j\} \,|\, \sigma_x = k, \tau > k] = E^x[\sum_{j=0}^{\infty} I\{S_j = x, \tau > j\}]$$

so that

$$\sum_{k=1}^{\infty} E^x[\sum_{j=\sigma_x}^{\infty} I\{S_j = x, \tau > j\} \,|\, \sigma_x = k, \tau > k]\; P^x\{\sigma_x = k, \tau > k\}$$

$$= \sum_{k=1}^{\infty} E^x[\sum_{j=0}^{\infty} I\{S_j = x, \tau > j\}]\; P^x\{\sigma_x = k, \tau > k\}$$

$$= E^x[\sum_{j=0}^{\infty} I\{S_j = x, \tau > j\}] \sum_{k=1}^{\infty} P^x\{\sigma_x = k, \tau > k\}$$

$$= G_A(x,x) P^x\{\tau > \sigma_x\}.$$

That is,

$$G_A(x,x) = 1 + G_A(x,x)P^x\{\tau > \sigma_x\}$$

which implies that

$$G_A(x,x) = \frac{1}{1 - P^x\{\tau > \sigma_x\}} = [P^x\{\tau < \sigma_x\}]^{-1}$$

noting that by definition $P^x\{\tau = \sigma_x\} = 0$.

Solution to Exercise (1.5.11). By Theorem 1.4.6, the unique function $f : \overline{A} \to R$ satisfying (a) and (b) is given by

$$f(x) = E^x[F(S_\tau) + \sum_{j=0}^{\tau-1} g(S_j)].$$

An alternative representation for f can be obtained by noticing that

$$E^x[F(S_\tau)] = \sum_{y \in \partial A} F(y) H_{\partial A}(x,y) \quad \text{and} \quad E^x[\sum_{j=0}^{\tau-1} g(S_j)] = \sum_{z \in A} g(z) G_A(x,z).$$

Indeed, the first equality follows since

$$E^x[F(S_\tau)] = \sum_{y \in \partial A} F(y) P^x\{S_\tau = y\} = \sum_{y \in \partial A} F(y) H_{\partial A}(x,y)$$

and the second follows since

$$E^x[\sum_{j=0}^{\tau-1} g(S_j)] = E^x[\sum_{j=0}^{\infty} g(S_j) I\{\tau > j\}] = \sum_{j=0}^{\infty} E^x[g(S_j) I\{\tau > j\}]$$

$$= \sum_{j=0}^{\infty} \sum_{z \in A} \sum_{k=0}^{\infty} g(z) I\{k > j\} P^x\{S_j = z, \tau = k\}.$$

But,

$$\sum_{j=0}^{\infty} \sum_{z \in A} \sum_{k=0}^{\infty} g(z) I\{k > j\} P^x\{S_j = z, \tau = k\} = \sum_{z \in A} g(z) \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} P^x\{S_j = z, \tau = k\}$$

$$= \sum_{z \in A} g(z) \sum_{j=0}^{\infty} P^x\{S_j = z, \tau > j\}$$

$$= \sum_{z \in A} g(z) G_A(x,z).$$

In summary,

$$f(x) = E^x[F(S_\tau) + \sum_{j=0}^{\tau-1} g(S_j)] = \sum_{y \in \partial A} F(y) H_{\partial A}(x, y) + \sum_{z \in A} g(z) G_A(x, z)$$

as required.