Stat 800 Spring 2006
Solutions to Assignment #2

Solution to Exercise (1.3.4). Let $a > 0$, and suppose that $\tau = \inf\{j \geq 0 : |S_j| \geq a\}$. Since the events $\{\sup_{1 \leq j \leq n} |S_j| \geq a\}$ and $\{\tau \leq n\}$ are equal, we will show that $P\{\tau \leq n\} \leq 2P\{|S_n| \geq a\}$. Elementary conditional probability gives

$$P\{|S_n| \geq a\} = P\{|S_n| \geq a|\tau \leq n\}P\{\tau \leq n\} + P\{|S_n| \geq a|\tau > n\}P\{\tau > n\} = P\{|S_n| \geq a|\tau \leq n\}P\{\tau \leq n\}$$

where the second equality follows since $P\{|S_n| \geq a|\tau > n\} = 0$ by the definition of τ. By symmetry and the strong Markov property we find

$$P\{|S_n| \geq a|\tau \leq n\} \geq \frac{1}{2}$$

from which we conclude

$$P\{|S_n| \geq a\} \geq \frac{1}{2}P\{\tau \leq n\}.$$

In other words,

$$P\{\sup_{1 \leq j \leq n} |S_j| \geq a\} = P\{\tau \leq n\} \leq 2P\{|S_n| \geq a\}$$

as required.

Solution to Exercise (1.4.2). The proof is virtually identical to the proof of Proposition 1.4.1. Assume $S_0 = x$. By the Markov property, $E(f(S_{n+1})|\mathcal{F}_n) = f(S_n) + \Delta f(S_n)$. If $B_n = \{\tau > n\}$, then $M_{n+1} = M_n$ on B_n^c and $E(M_{n+1}|\mathcal{F}_n) = (f(S_n) + \Delta f(S_n))I_{B_n} + M_n I_{B_n^c}$. Since f is superharmonic, it follows that $\Delta f(S_n) \leq 0$ on B_n which gives

$$E(M_{n+1}|\mathcal{F}_n) \leq I_{B_n} f(S_n) + I_{B_n^c} M_n = M_n$$

so that M_n is a supermartingale with respect to $\mathcal{F}_n$.

Solution to Exercise (1.4.3). To begin, it is clear that M_n is $\mathcal{F}_n$-measurable, and that $E(|M_n|) \leq E(|S_n|^2) + n < \infty$ for each n. Since S_n is a d-dimensional simple random walk, we write $S_n = (S_n^1, \ldots, S_n^d)$ (and note that $S_n^1, \ldots, S_n^d$ are *not* independent one-dimensional simple random walks on Z). We also write $X_n = (X_n^1, \ldots, X_n^d)$ so that

$$S_{n+1} = S_n + X_{n+1} = (S_n^1 + X_{n+1}^1, \ldots, S_n^d + X_{n+1}^d).$$

(Now, however, note that S_n^i and X_{n+1}^i *are* independent for each $i = 1, \ldots, d$.) This gives

$$|S_{n+1}|^2 = \sum_{j=1}^d (S_{n+1}^j)^2 = \sum_{j=1}^d (S_n^j + X_{n+1}^j)^2 = \sum_{j=1}^d (S_n^j)^2 + (X_{n+1}^j)^2 + 2S_n^j X_{n+1}^j = |S_n|^2 + |X_{n+1}|^2 + 2\sum_{j=1}^d S_n^j X_{n+1}^j$$

and so

$$E(|S_{n+1}|^2|\mathcal{F}_n) = E(|S_n|^2|\mathcal{F}_n) + E(|X_{n+1}|^2|\mathcal{F}_n) + 2\sum_{j=1}^d E(S_n^j X_{n+1}^j|\mathcal{F}_n).$$

Since S_n is $\mathcal{F}_n$-measurable, and since X_{n+1} is independent of $\mathcal{F}_n$, we use properties of conditional expectation to conclude that $E(|S_n|^2|\mathcal{F}_n) = |S_n|^2$ and $E(|X_{n+1}|^2|\mathcal{F}_n) = E(|X_{n+1}|^2) = 1$. Furthermore, $E(S_n^j X_{n+1}^j|\mathcal{F}_n) = 0$ for each $j = 1, \ldots, d$. Indeed, since S_n^j is $\mathcal{F}_n$-measurable, and X_{n+1}^j is independent of $\mathcal{F}_n$, it follows from properties of conditional expectation that $E(S_n^j X_{n+1}^j|\mathcal{F}_n) = S_n^j E(X_{n+1}^j|\mathcal{F}_n) = S_n^j E(X_{n+1}^j) = 0$. Hence,

$$E(|S_{n+1}|^2|\mathcal{F}_n) = |S_n|^2 + 1$$

so that

$$E(M_{n+1}|\mathcal{F}_n) = E(|S_{n+1}|^2 - (n+1)|\mathcal{F}_n) = |S_n|^2 + 1 - (n+1) = |S_n|^2 - n = M_n$$

showing M_n is a martingale with respect to $\mathcal{F}_n$.

Exercise. The purpose of this exercise is to give the Brownian motion analogue of Lemma 1.5.1. Suppose that B_t is a standard d-dimensional Brownian motion. Show that for any $a > 0$, there exists $c_a < \infty$ such that for all t, $\rho > 0$,

$$P\{|B_t| \geq a\rho t^{1/2}\} \leq c_a e^{-\rho}.$$

Solution. We begin by writing $B_t = (B_t^1, \ldots, B_t^d)$ where $B_t^1, \ldots, B_t^d$ are independent (standard) one-dimensional Brownian motions. Therefore,

$$\begin{aligned} P\{|B_t| \geq a\rho t^{1/2}\} = P\{|B_t|^2 \geq a^2\rho^2 t\} = P\{(B_t^1)^2 + \cdots + (B_t^d)^2 \geq a^2\rho^2 t\} &\leq d\,P\{(B_t^1)^2 \geq d^{-1}a^2\rho^2 t\} \\ &= d\,P\{|B_t^1| \geq d^{-1/2}a\rho t^{1/2}\}. \end{aligned}$$

By symmetry,

$$P\{|B_t^1| \geq d^{-1/2}a\rho t^{1/2}\} = 2P\{B_t^1 \geq d^{-1/2}a\rho t^{1/2}\}$$

and by Brownian scaling,

$$P\{B_t^1 \geq d^{-1/2}a\rho t^{1/2}\} = P\{B_1^1 \geq d^{-1/2}a\rho\}$$

since $t^{-1/2}B_t^1 \sim B_1 \sim N(0,1)$. Chebychev's inequality then yields

$$P\{B_1^1 \geq d^{-1/2}a\rho\} = P\{d^{1/2}a^{-1}B_1^1 \geq \rho\} \leq e^{-\rho}E[\exp\{d^{1/2}a^{-1}B_1^1\}].$$

The explicit form of the moment generating function of a $N(0,1)$ random variable gives

$$E[\exp\{d^{1/2}a^{-1}B_1^1\}] = \exp\{\frac{d}{2a^2}\}.$$

Combining everything, we therefore find

$$P\{|B_t| \geq a\rho t^{1/2}\} = 2dP\{d^{1/2}a^{-1}B_1^1 \geq \rho\} \leq \exp\{\frac{d}{2a^2}\}e^{-\rho} = c_a e^{-\rho}$$

as required.