Stat 800 Spring 2006
Solutions to Assignment #1

Exercise. Derive equation (1.5).

Solution. Using the definitions of $p_{m+n}(x,y)$ and $\tilde{S}_n$, we find

$$p_{m+n}(x,y) = P^x\{S_{m+n} = y\} = P^x\{\tilde{S}_n = y - S_m\}.$$

Conditioning on the value of S_m gives

$$P^x\{\tilde{S}_n = y - S_m\} = \sum_{z \in Z^d} P^x\{\tilde{S}_n = y - z | S_m = z\} P^x\{S_m = z\}.$$

Since $\tilde{S}_n$ is a simple random walk starting at 0 and independent of $\{X_1, \ldots, X_m\}$ (in particular, $\tilde{S}_n$ is independent of S_m), we see that

$$P^x\{\tilde{S}_n = y - z | S_m = z\} = P\{\tilde{S}_n = y - z\} = p_n(y - z).$$

Combining everything gives

$$p_{m+n}(x,y) = \sum_{z \in Z^d} P\{\tilde{S}_n = y - z\} P^x\{S_m = z\} = \sum_{z \in Z^d} p_n(y-z) p_m(x,z) = \sum_{z \in Z^d} p_m(x,z) p_n(z,y)$$

since $p_n(y - z) = p_n(z, y)$ by (1.2).

Exercise. Show that τ is a stopping time with respect to $\mathcal{G}_n$ if and only if $\{\tau \le n\} \in \mathcal{G}_n$ for every n.

Solution. If τ is a stopping time with respect to $\mathcal{G}_n$, then by definition $\{\tau = n\} \in \mathcal{G}_n$. Since $\mathcal{G}_n$ is filtration (so that $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$) we find $\{\tau = j\} \in \mathcal{G}_j \subset \mathcal{G}_n$ for all $j = 1, 2, \ldots, n$. Therefore, since $\mathcal{G}_n$ is a σ-algebra, we conclude

$$\{\tau \le n\} = \bigcup_{j=1}^{n} \{\tau = j\} \in \mathcal{G}_n.$$

On the other hand, suppose that $\{\tau \le n\} \in \mathcal{G}_n$ for every n. Therefore, since $\mathcal{G}_{n-1}$ is a σ-algebra and since $\mathcal{G}_{n-1} \subset \mathcal{G}_n$, it follows that $\{\tau \le n-1\}^c \in \mathcal{G}_{n-1} \subset \mathcal{G}_n$. We then conclude

$$\{\tau = n\} = \{\tau \le n\} \cap \{\tau \le n-1\}^c \in \mathcal{G}_n$$

so that τ is a stopping time with respect to $\mathcal{G}_n$.

Exercise. Show that if τ_1 and τ_2 are both stopping times with respect to $\mathcal{G}_n$, then so too are $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$.

Solution. Using the result of the previous exercise, to show that $\tau_1 \wedge \tau_2$ is a stopping time with respect to $\mathcal{G}_n$, it suffices to show that $\{\tau_1 \wedge \tau_2 \le n\} \in \mathcal{G}_n$. Since τ_1 and τ_2 are stopping times, we have $\{\tau_1 > n\} \in \mathcal{G}_n$ and $\{\tau_2 > n\} \in \mathcal{G}_n$ so that $\{\tau_1 \wedge \tau_2 > n\} = \{\tau_1 > n\} \cap \{\tau_2 > n\} \in \mathcal{G}_n$. Therefore, $\{\tau_1 \wedge \tau_2 \le n\} = \{\tau_1 \wedge \tau_2 > n\}^c \in \mathcal{G}_n$. Similarly, to show that $\tau_1 \vee \tau_2$ is a stopping time with respect to $\mathcal{G}_n$, it suffices to show that $\{\tau_1 \vee \tau_2 \le n\} \in \mathcal{G}_n$. Since τ_1 and τ_2 are stopping times, we have $\{\tau_1 \le n\} \in \mathcal{G}_n$ and $\{\tau_2 \le n\} \in \mathcal{G}_n$ so that $\{\tau_1 \vee \tau_2 \le n\} = \{\tau_1 \le n\} \cap \{\tau_2 \le n\} \in \mathcal{G}_n$.

Solution to Exercise (1.3.1). Suppose that τ is a stopping time with respect to $\mathcal{G}_n$. To show that $\mathcal{G}_\tau$ is a σ-algebra, it suffices to show that the three conditions in the definition of σ-algebra are satisfied, namely **(i)** $\emptyset \in \mathcal{G}_\tau$, **(ii)** if $A_i \in \mathcal{G}_\tau$ for $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}_\tau$, and **(iii)** if $A \in \mathcal{G}_\tau$, then $A^c \in \mathcal{G}_\tau$.

(i) Since $\mathcal{G}_n$ is a σ-algebra for each n, we know $\emptyset \in \mathcal{G}_n$. Therefore, $\emptyset \cap \{\tau \leq n\} = \emptyset \in \mathcal{G}_n$ for each n, so that $\emptyset \in \mathcal{G}_\tau$.

(ii) Suppose that $A_i \in \mathcal{G}_\tau$ for $i = 1, 2, \ldots$, so that $A_i \cap \{\tau \leq n\} \in \mathcal{G}_n$ for each n. Since $\mathcal{G}_n$ is a σ-algebra, $\bigcup_{i=1}^{\infty} \{A_i \cap \{\tau \leq n\}\} \in \mathcal{G}_n$. Therefore, $\bigcup_{i=1}^{\infty} \{A_i \cap \{\tau \leq n\}\} = \{\bigcup_{i=1}^{\infty} A_i\} \cap \{\tau \leq n\} \in \mathcal{G}_n$ for each n so that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}_\tau$.

(iii) Notice that $\{\tau \leq n\} \in \mathcal{G}_n$ for each n since τ is a stopping time with respect to $\mathcal{G}_n$. Since $\mathcal{G}_n$ is a σ-algebra, $\{\tau \leq n\}^c = \{\tau > n\} \in \mathcal{G}_n$. Suppose that $A \in \mathcal{G}_\tau$ so that $A \cap \{\tau \leq n\} \in \mathcal{G}_n$ for each n. Consequently, $[A \cap \{\tau \leq n\}] \cup \{\tau > n\} = A \cup \{\tau > n\} \in \mathcal{G}_n$. Again, as $\mathcal{G}_n$ is a σ-algebra, it follows that $[A \cup \{\tau > n\}]^c = A^c \cap \{\tau \leq n\} \in \mathcal{G}_n$, so that $A^c \in \mathcal{G}_\tau$.