Using multiple SLE to explain a certain observable in the 2d Ising model

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SLE-CFT

The Schramm-Loewner evolution with parameter κ (SLE$_\kappa$) was introduced in 1999 by Oded Schramm while considering possible scaling limits of loop-erased random walk.

Since then, it has successfully been used to study a number of lattice models from two-dimensional statistical mechanics including percolation, uniform spanning trees, self-avoiding walk, and the Ising model.

In general, there is some understanding of how SLE can be used to formalize two-dimensional conformal field theory, but nevertheless there is still a lot of work to be done.

SLE-CFT (cont)

Conformal field theory (CFT) relies on the concept of a local field and its correlations in order to generate predictions about the model under consideration.

Briefly, in CFT, the central charge $\mathbf{c}$ plays a key role in delimiting the universality classes of a variety of lattice model scaling limits.

We now know that the SLE parameter κ and the central charge $\mathbf{c}$ are related through

$$\mathbf{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}.$$

The Ising Model

The Ising model is, perhaps, the simplest interacting many particle system in statistical mechanics. Although it had its origins in magnetism, it is now of importance in the context of phase transitions.

Suppose that $D \subset \mathbb{C}$ is a bounded, simply connected domain with Jordan boundary.

Consider the discrete approximation given by $D \cap \mathbb{Z}^2$.

Assign to each vertex of the square lattice a spin — either up $(+1)$ or down (-1).

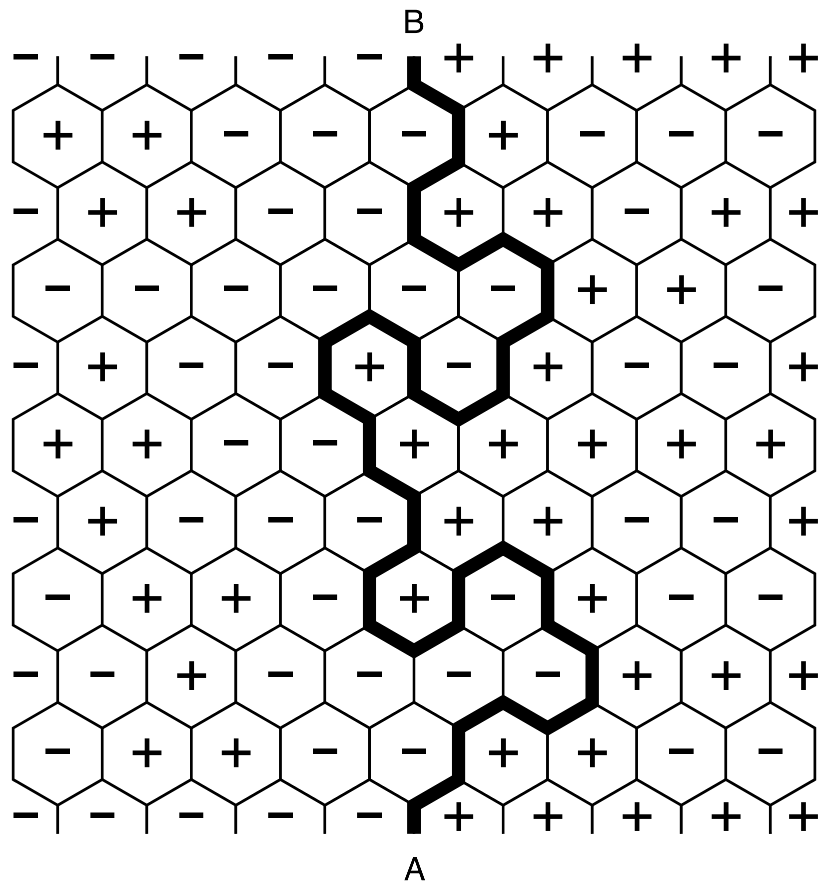
Let ω denote a configuration of spins; i.e., an element of $\Omega = \{-1, +1\}^N$ where N is the number of vertices.

Associate to the configuration the Hamiltonian

$$H(\omega) = -\sum_{i \sim j} \sigma_i \sigma_j$$

where the sum is over all nearest neighbours and $\sigma_i \in \{-1, +1\}$.

The Ising Model (cont)



The Ising Model (cont)

Define a probability measure

$$P(\{\omega\}) = \frac{\exp\{-\beta H(\omega)\}}{Z}$$

where $\beta > 0$ is a parameter and

$$Z = \sum_{\omega} \exp\{-\beta H(\omega)\}$$

is the partition function (or normalizing constant).

The parameter β is the inverse-temperature $\beta = 1/T$. It is known that there is a critical temperature T_c which separates the ferromagnetic ordered phase (below T_c) from the paramagnetic disordered phase (above T_c).

Furthermore, many physical properties (i.e., observables), such as the thermodynamic free energy, entropy, magnetization, and spin-spin correlation can be determined from the partition function.

The Ising Model (cont)

Traditionally, scaling limits in CFT are described by critical exponents.

For example, the spin-spin correlation

$$\langle \sigma_i, \sigma_j \rangle = \sum_{\omega} \sigma_i \sigma_j P(\{\omega\}) \sim \frac{\exp\{-|i-j|/\xi\}}{|i-j|^{\eta}}$$

where the correlation length ξ scales like

$$\xi \sim |T - T_c|^{-\nu}.$$

At T_c, the correlation length ξ diverges, the Ising model becomes scale invariant, and we have

$$\langle \sigma_i, \sigma_j \rangle \sim |i-j|^{-\eta}.$$

The Ising Model (cont)

The point-of-view introduced by SLE is that of an interface.

Consider fixing two arcs on the boundary of the domain and holding one boundary arc all at spin up and the other all at spin down.

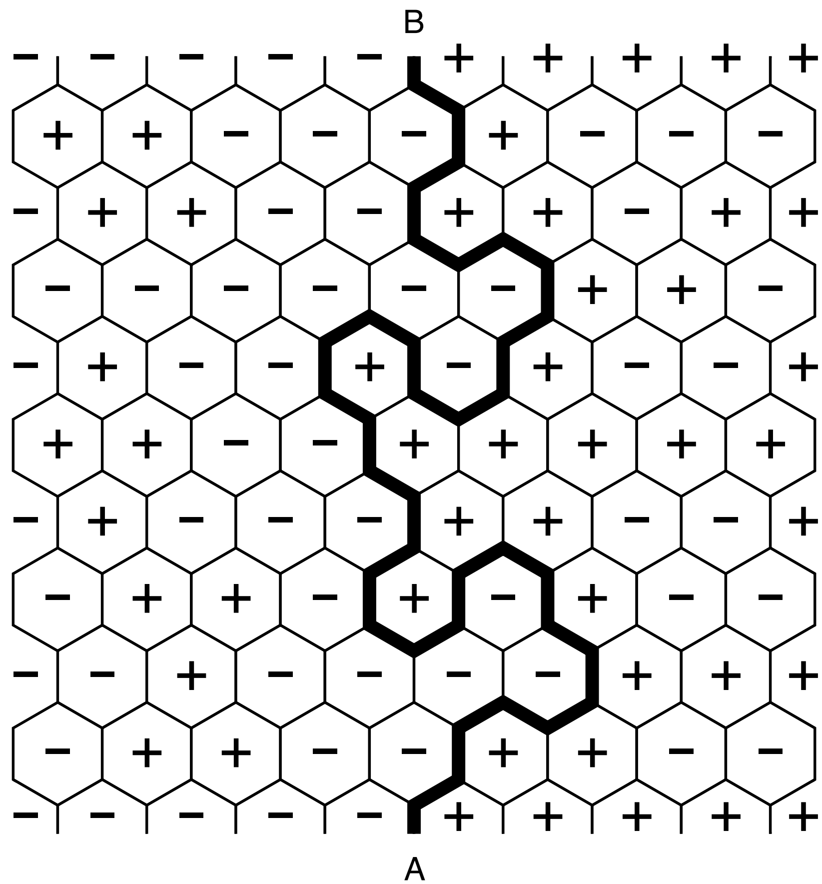
$P(\{\omega\})$ now induces a probability measure on curves (interfaces) connecting the two boundary points where the boundary conditions change.

S. Smirnov has recently showed that as the lattice spacing shrinks to 0, the interfaces converge to SLE_3.

Formally, let (D, z, w) be a simply connected Jordan domain with distinguished boundary points z and w. Let $D_n = \frac{1}{n}\mathbb{Z}^2 \cap D$ denote the $1/n$-scale square lattice approximation of D, and let z_n, w_n be the corresponding boundary points of D_n, i.e., we need $(D_n, z_n, w_n) \to (D, z, w)$ in the Caratheodory sense as $n \to \infty$.

If $P_n = P_n(D_n, z_n, w_n)$ denotes the law of the discrete interface, then P_n converges weakly to $\mu_{D,z,w}$, the law of chordal SLE_3 in D from z to w.

The Ising Model (cont)



Multiple Interfaces in the Ising Model

"Though one can argue whether the scaling limits of interfaces in the Ising model are of physical relevance, their identification opens possibility for computation of correlation functions and other objects of interest in physics." (Smirnov, 2007)

Consider four distinct points z_1, z_2, z_3, z_4 ordered counterclockwise around ∂D. Alternate the boundary conditions between plus and minus, changing at each z_i.

Sample the Ising model at cirticality on D. There will now be two interfaces, either (I) joining $z_1 \leftrightarrow z_4$ and $z_2 \leftrightarrow z_3$, OR (II) joining $z_1 \leftrightarrow z_2$ and $z_3 \leftrightarrow z_4$.

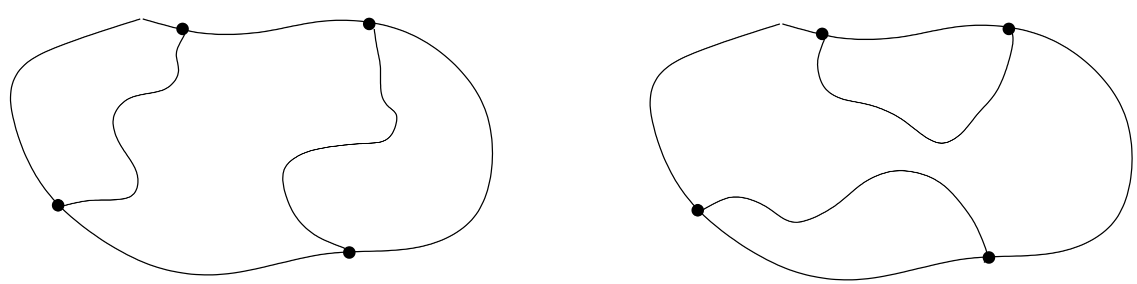


Fig: The two possible configuration-types corresponding to four distinguished boundary points.

Multiple Interfaces in the Ising Model (cont)

Question: What is the probability that the resulting crossings are of Type I?

Answer: In the discrete case, it is

$$\frac{Z_I}{Z_I + Z_{II}}$$

where Z_I denotes the partition function corresponding to all possible configurations having a crossing of Type I.

Using SLE, we can compute the limit of this probability as the lattice spacing shrinks to 0.

This crossing probability is the non-local observable considered by Arguin and Saint-Aubin, and in more generality by Bauer, Bernard, and Kytölä.

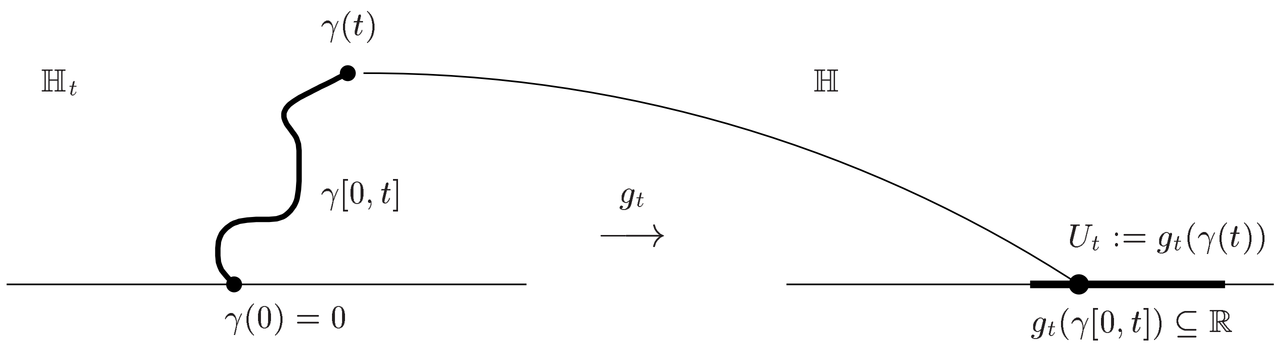
Review of SLE

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half plane, and consider a simple (non-self-intersecting) curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$.

For every fixed $t \geq 0$, the slit plane $\mathbb{H}_t := \mathbb{H} \setminus \gamma(0, t]$ is simply connected and so by the Riemann mapping theorem, there exists a unique conformal transformation $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$ which can be expanded as

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty,$$

where $b(t) = \text{hcap}(\gamma(0, t])$ is the **half-plane capacity** of γ up to time t.



It can be shown that there is a unique point $U_t \in \mathbb{R}$ for all $t \geq 0$ with $U_t := g_t(\gamma(t))$ and that the function $t \mapsto U_t$ is continuous.

Review of SLE (cont)

$$g_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty, \quad \mathbb{H}_t = \mathbb{H} \setminus \gamma(0,t]$$

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{H}_t \to \mathbb{H}$, can be described by a PDE involving U_t.

This is due to C. Loewner (1923) who showed that if γ is a curve as above such that its half-plane capacity $b(t)$ is C^1 and $b(t) \to \infty$ as $t \to \infty$, then for $z \in \mathbb{H}$ with $z \notin \gamma[0,\infty)$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy the PDE

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Note that if $b(t) \in C^1$ is an increasing function, then we can reparametrize the curve γ so that $\mathrm{hcap}(\gamma(0,t]) = b(t)$. This is the so-called **parametrization by capacity**.

Review of SLE (cont)

$$\frac{\partial}{\partial t} g_t(z) = \frac{\dot{b}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{$*$}$$

The obvious thing to do now is to start with a continuous function $t \mapsto U_t$ from $[0, \infty)$ to $\mathbb{R}$ and solve the Loewner equation for g_t.

Ideally, we would like to solve $(*)$ for g_t, define simple curves $\gamma(t)$, $t \geq 0$, by setting $\gamma(t) = g_t^{-1}(U_t)$, and have g_t map $\mathbb{H} \setminus \gamma(0, t]$ conformally onto $\mathbb{H}$.

Although this is the intuition, it is not quite precise because we see from the denominator on the right-side of $(*)$ that problems can occur if $g_t(z) - U_t = 0$.

Formally, if we let T_z be the supremum of all t such that the solution to $(*)$ is well-defined up to time t with $g_t(z) \in \mathbb{H}$, and we define $\mathbb{H}_t = \{z : T_z > t\}$, then g_t is the unique conformal transformation of $\mathbb{H}_t$ onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $t \to \infty$.

Review of SLE (cont)

The novel idea of Schramm was to take the continuous function U_t to be a one-dimensional Brownian motion starting at 0 with variance parameter $\kappa \geq 0$.

The **chordal Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard parametrization** (or simply SLE$_\kappa$) is the random collection of conformal maps $\{g_t,\, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t}\, g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}\, W_t}, \quad g_0(z) = z, \tag{LE}$$

where W_t is a standard one-dimensional Brownian motion.

Review of SLE (cont)

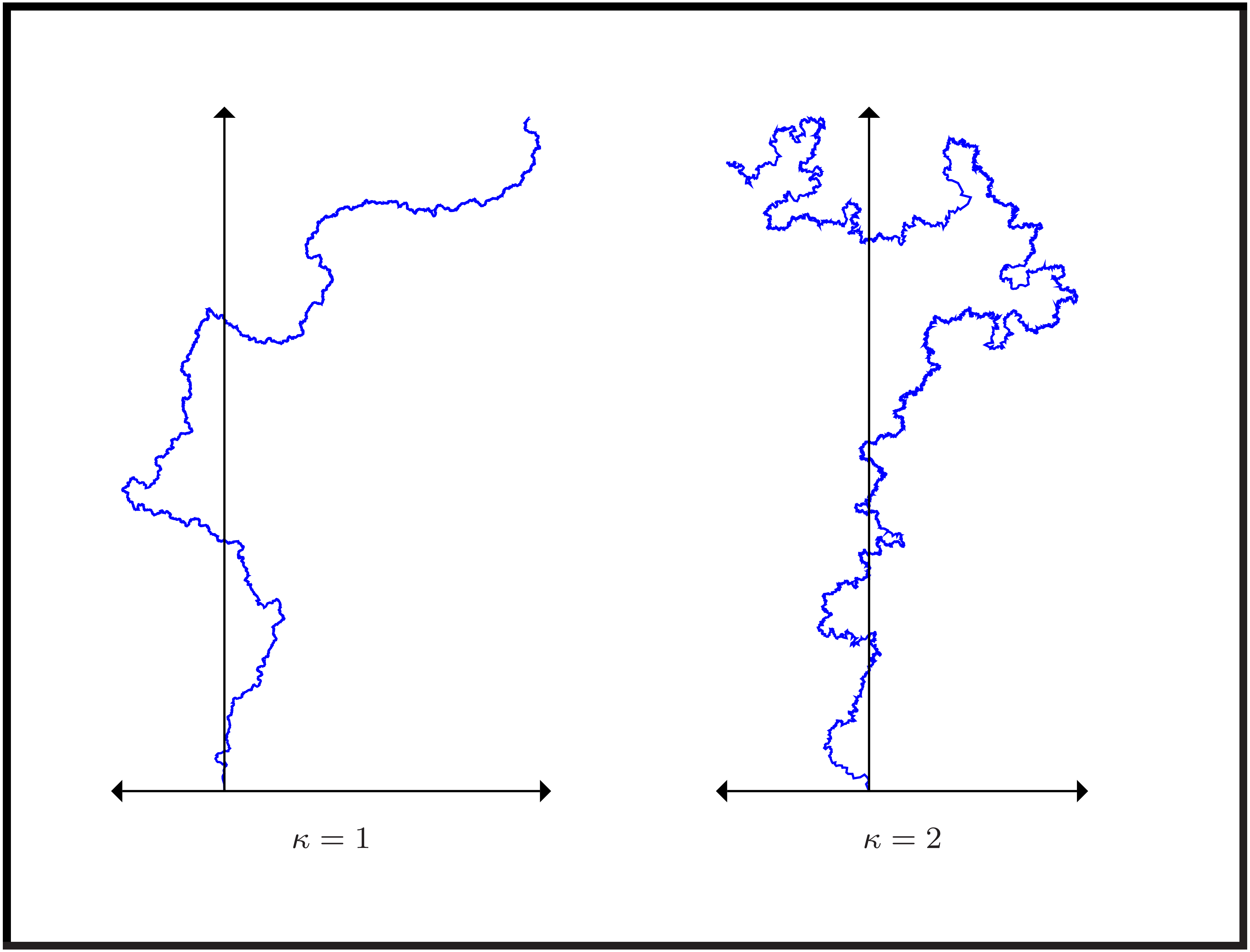
The question is now whether there exists a curve associated with the maps g_t.

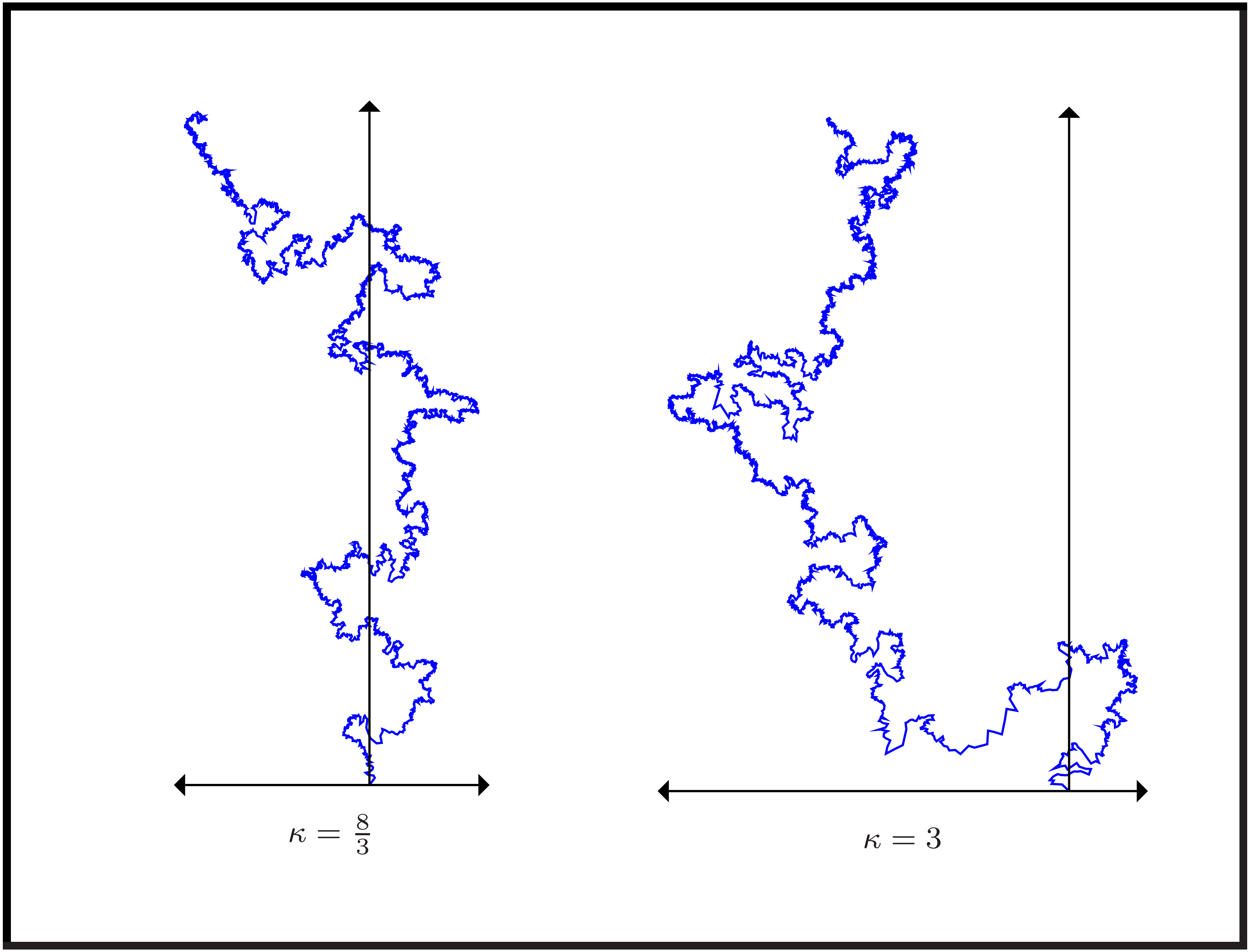
- If $0 < \kappa \leq 4$, then there exists a random simple curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset \mathbb{H}$, i.e., the curve $\gamma(t) = g_t^{-1}(\sqrt{\kappa} B_t)$ never re-visits $\mathbb{R}$. As well, the maps g_t obtained by solving (∗) are conformal transformations of $\mathbb{H} \setminus \gamma(0, t]$ onto $\mathbb{H}$. For this range of κ, our intuition matches the theory!

- For $4 < \kappa < 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$. These curves have double points and they do hit $\mathbb{R}$, but they never cross themselves! As such, $\mathbb{H} \setminus \gamma(0, t]$ is not simply connected. However, $\mathbb{H} \setminus \gamma(0, t]$ does have a unique connected component containing ∞. This is $\mathbb{H}_t$ and the maps g_t are conformal transformations of $\mathbb{H}_t$ onto $\mathbb{H}$. We think of $\mathbb{H}_t = \mathbb{H} \setminus K_t$ where K_t is the hull of $\gamma(0, t]$ visualized by taking $\gamma(0, t]$ and filling in the holes.

- For $\kappa \geq 8$, there exists a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ which is space-filling! Furthermore, it has double points, but does not cross itself!

As a result, we also refer to the curve γ as chordal SLE$_\kappa$. SLE paths are extremely rough: the Hausdorff dimension of a chordal SLE$_\kappa$ path is $\min\{1 + \kappa/8, 2\}$.


$\kappa = 1$
$\kappa = 2$


$\kappa = \frac{8}{3}$
$\kappa = 3$

Review of SLE (cont)

Since there exists a curve γ associated with the maps g_t, it is possible to reparametrize it.

It can be shown that if U_t is a standard one-dimensional Brownian motion, then the solution to the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2/\kappa}{g_t(z) - U_t} = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

is chordal SLE$_\kappa$ parametrized so that $\mathrm{hcap}(\gamma(0,t]) = 2t/\kappa = at$.

Finally, chordal SLE as we have defined it can also be thought of as a measure on paths in the upper half plane $\mathbb{H}$ connecting the boundary points 0 and ∞.

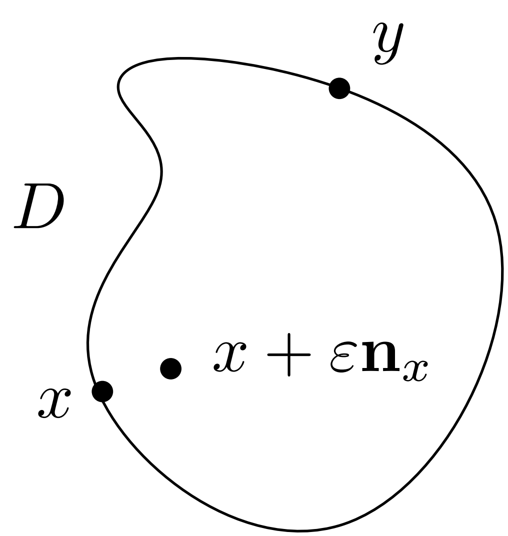
SLE is conformally invariant and so we can define chordal SLE$_\kappa$ in any simply connected domain D connecting distinct boundary points z and w to be the image of chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ under a conformal transformation from $\mathbb{H}$ onto D sending $0 \mapsto z$ and $\infty \mapsto w$.

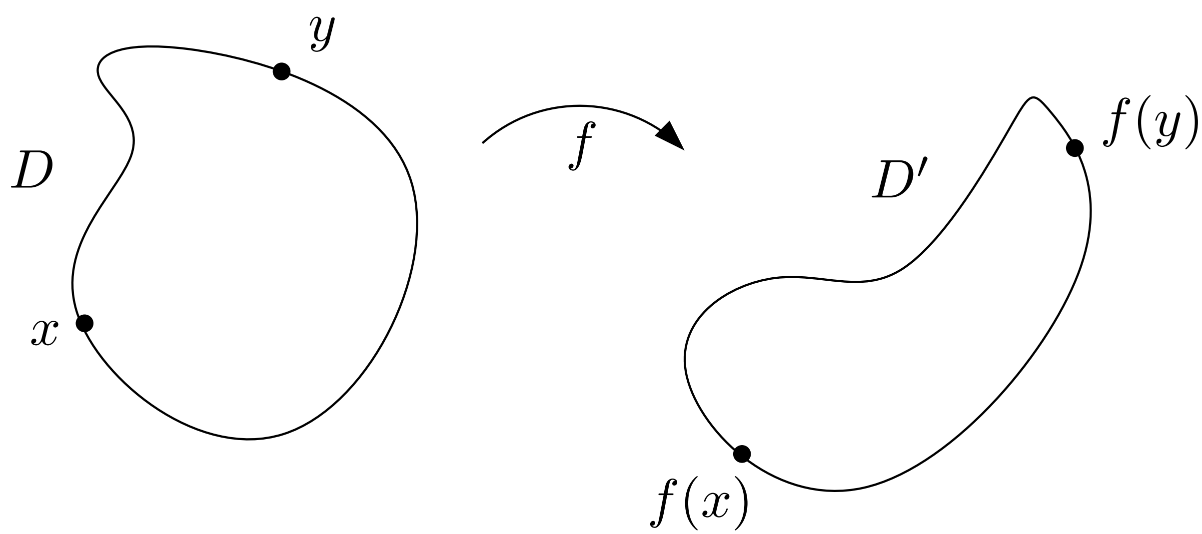
Excursion Poisson Kernel

Suppose that $D \subset \mathbb{C}$ is a simply connected Jordan domain and that ∂D is locally analytic at x and y. The excursion Poisson kernel is defined as

$$H_{\partial D}(x,y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} H_D(x + \varepsilon \mathbf{n}_x, y)$$

where $H_D(z,y)$ for $z \in D$ is the usual Poisson kernel, and $\mathbf{n}_x$ is the unit normal at x pointing into D.





Proposition: If $f : D \to D'$ is a conformal transformation where $D' \subset \mathbb{C}$ is also a simply connected Jordan domain, and $\partial D'$ is locally analytic at $f(x)$, $f(y)$, then

$$H_{\partial D}(x, y) = |f'(x)||f'(y)|H_{\partial D'}(f(x), f(y)).$$

Example: Unit disk $\mathbb{D}$: $H_{\partial \mathbb{D}}(x, y) = \dfrac{1}{\pi\,|y - x|^2} = \dfrac{1}{2\pi(1 - \cos(\arg y - \arg x))}$.

Example: Upper half plane $\mathbb{H}$: $H_{\partial \mathbb{H}}(x, y) = \dfrac{1}{\pi\,(y - x)^2}$.

A Finite Measure on SLE Paths

Let $\mu_D(z,w)$ denote the chordal SLE$_\kappa$ probability measure on paths in D from z to w.

Define the finite measure

$$Q_D(z,w) = H_D(z,w)\mu_D(z,w)$$

where $H_D(z,w)$ is defined for the upper half plane $\mathbb{H}$ by setting

$$H_{\mathbb{H}}(0,\infty) = 1 \quad \text{and} \quad H_{\mathbb{H}}(x,y) = \frac{1}{|y-x|^{2b}}$$

and for other simply connected domains D by conformal covariance

$$H_D(z,w) = |f'(z)|^b \, |f'(w)|^b H_{D'}(f(z), f(w))$$

where $f : D \to D'$ is a conformal transformation (assuming appropriate smoothness) and $b > 0$ is a parameter.

A Finite Measure on SLE Paths (cont)

If we choose $b = \frac{6-\kappa}{2\kappa}$, then for $b \geq \frac{1}{4}$ (i.e., $0 < \kappa \leq 4$), the measure $Q_D(z,w)$ satisfies:

- **Conformal covariance.** If $f : D \to f(D)$ is a conformal transformation and $f(D)$ is analytic at $f(\mathbf{z})$, $f(\mathbf{w})$, then

$$f \circ Q_D(z,w) = |f'(z)|^b \, |f'(w)|^b \, Q_{f(D)}(f(z), f(w))$$

- **Boundary perturbation.** If $D \subset D'$ and ∂D, $\partial D'$ agree near z, w, then

$$Y_{D,D'}(z,w)(\gamma) = \frac{dQ_D(z,w)}{dQ_{D'}(z,w)}(\gamma) = 1\{\gamma \subset D\} e^{\mathbf{c}\Theta/2}$$

  where Θ is the measure of the set of Brownian loops in D' that intersect both γ and D, and $\mathbf{c} = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$.

- In particular, if $f : D' \to f(D')$ is a conformal transformation, then

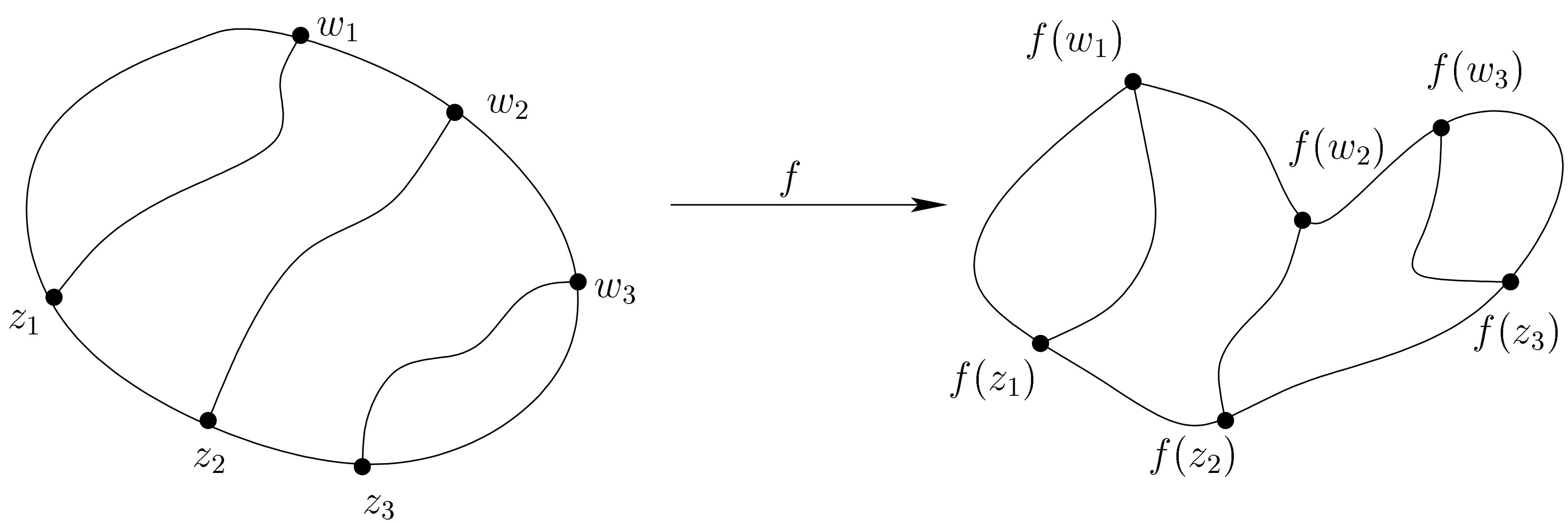
$$\frac{dQ_D(z,w)}{dQ_{D'}(z,w)} = \frac{dQ_{f(D)}(f(z), f(w))}{dQ_{f(D')}(f(z), f(w))}.$$

A Finite Measure on Multiple SLE Paths

Let $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$ denote n-tuples of distinct points in ∂D ordered counterclockwise.

Goal: To construct a finite measure $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n-tuples of mutually avoiding simple curves with γ_i connecting z_i to w_i.

This measure should satisfy conformal covariance, boundary perturbation, and a cascade relationship.



A Finite Measure on Multiple SLE Paths (cont)

$Q_{D,b,n}(\mathbf{z},\mathbf{w})$, the n-path SLE$_\kappa$ measure in D, is defined to be the measure that is absolutely continuous with respect to the product measure

$$Q_{D,b}(z_1,w_1)\times\cdots\times Q_{D,b}(z_n,w_n)$$

with Radon-Nikodym derivative $Y(\bar\gamma)=Y_{D,b,\mathbf{z},\mathbf{w}}(\gamma^1,\ldots,\gamma^n)$ given by

$$Y(\bar\gamma)=1\{\gamma^k\cap\gamma^l=\emptyset,\ 1\le k<l\le n\}\ \exp\left\{\frac{\mathbf{c}}{2}\sum_{k=1}^{n-1}\Theta(D;\gamma^k,\gamma^{k+1})\right\}$$

where $\Theta(D;V_1,V_2)$ is the Brownian loop measure of loops in D intersecting both V_1 and V_2.

If $\mathbf{c}\le 1$, it can be shown that $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ is a finite measure.

Existence of the Configurational Measure

Theorem (K-Lawler, 2007): *For any $b \geq \frac{1}{4}$, there exists a family of measures $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n-tuples of mutually avoiding simple curves satisfying*

- *conformal covariance,*
- *boundary perturbation,*
- *cascade relation,*
- *Markov property.*

Moreover, the simple curve γ^i is a chordal SLE$_\kappa$ from z_i to w_i in D where

$$\kappa = \frac{6}{2b+1} \longleftrightarrow b = \frac{6-\kappa}{2\kappa}.$$

Note: $b \geq \frac{1}{4} \longleftrightarrow 0 < \kappa \leq 4$

Note: These four properties were not discovered accidentally. We were told by CFT what properties the measure had to satisfy, and what the relationship between all the parameters had to be.

The Partition Function for Two Paths

Define $H_{D,b,n}(\mathbf{z},\mathbf{w})$ to be the mass of the measure $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ and note that $H_{D,b,n}$ satisfies the scaling rule

$$H_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b \, |f'(\mathbf{w})|^b \, H_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})).$$

Here $|f'(\mathbf{z})| = |f'(z_1)| \cdots |f'(z_n)|$.

Furthermore, if we define

$$\tilde{H}_{D,b,n}(\mathbf{z},\mathbf{w}) = \frac{H_{D,b,n}(\mathbf{z},\mathbf{w})}{H_{D,b}(z_1,w_1)\cdots H_{D,b}(z_n,w_n)},$$

then this is a conformal invariant.

The Partition Function for Two Paths (cont)

By conformal invariance, it suffices to work in $D = \mathbb{H}$. Let $0 < x < y < \infty$.

Proposition: If $b \geq 1/4$, then

$$\tilde{H}_{\mathbb{H},b,2}((0,x),(\infty,y)) = \frac{\Gamma(2a)\,\Gamma(6a-1)}{\Gamma(4a)\,\Gamma(4a-1)}\,(x/y)^a\,F(2a, 1-2a, 4a; x/y).$$

where $F = {}_2F_1$ denotes the hypergeometric function and $a = \dfrac{2}{\kappa} = \dfrac{2b+1}{3}$.

Note: This result first appeared rigorously in J. Dubédat, and was derived using CFT by M. Bauer, D. Bernard, and K. Kytölä. Our configurational approach provides another rigorous derivation.

Note: As we will see in a moment, the special case of the Ising model actually appeared earlier in L.-P. Arguin and Y. Saint-Aubin.

Note: Although our construction is restricted to simple curves ($0 < \kappa \leq 4$), if we formally plug in $\kappa = 6$, then we recover Cardy's formula for percolation.

The Partition Function for Two Paths (cont)

The proof of this proposition is accomplished by deriving and then solving a differential equation satisfied by $\tilde{H}_{\mathbb{H},b,2}((0,x),(\infty,y))$.

By scaling, $\tilde{H}_{\mathbb{H},b,2}((0,x),(\infty,y)) = \phi(x/y)$ for some function $\phi = \phi_{\mathbb{H},b}$ of one variable.

It can be shown that the ODE satisfied by ϕ is

$$u^2\,(1-u)^2\,\phi''(u) + 2\,u\,(a - u + (1-a)\,u^2)\,\phi'(u) - a(3a-1)(1-u)^2\,\phi(u) = 0.$$

In the case that $\kappa = 3$ (so that $a = 2/3$), this differential equation reduces to

$$3u^2\,(1-u)\,\phi''(u) + 2\,u\,(2-u)\,\phi'(u) - 2(1-u)\,\phi(u) = 0.$$

Let $g(z) = \phi(1-z)$ so that g satisfies

$$3z(z-1)^2\,g''(z) + 2\,(z-1)\,(z+1)\,g'(z) - 2z\,g(z) = 0.$$

The Partition Function for Two Paths (cont)

For the Ising model, note that

$$\kappa = 3, \quad a = \frac{2}{3}, \quad b = \frac{1}{2}, \quad \mathbf{c} = \frac{1}{2}, \quad d = \frac{11}{8}.$$

Also, recall that

$$3z(z-1)^2\, g''(z) + 2\,(z-1)\,(z+1)\, g'(z) - 2z\, g(z) = 0.$$

This differential equation is exactly the one that was derived by L.-P. Arguin and Y. Saint-Aubin in 2002 using techniques from conformal field theory in order to obtain theoretical predictions for the behaviour of the crossing probability (i.e., the non-local observable for the 2-D Ising model.)

For Arguin and Saint-Aubin, the function g was, basically, the "four-point correlation function of the local field of conformal weight $1/2$."

Calculating the Crossing Probability

By conformal invariance, it is enough to work in the upper half plane $\mathbb{H}$, with boundary points 0, 1, ∞, and x where $0 < x < 1$ is a real number.

The possible interface configurations are therefore of two types, namely (I) a simple curve connecting 0 to ∞ and a simple curve connecting x to 1, or (II) a simple curve connecting 0 to x and a simple curve connecting 1 to ∞.

The configurational measure corresponding to Type I is

$$Q_{\mathbb{H},b,2}((0,x),(1,\infty))$$

and the configurational measure corresponding to Type II is

$$Q_{\mathbb{H},b,2}((x,1),(\infty,0)).$$

By symmetry, however,

$$Q_{\mathbb{H},b,2}((x,1),(\infty,0)) = Q_{\mathbb{H},b,2}((0,1-x),(1,\infty)).$$

Calculating the Crossing Probability (cont)

Therefore, the partition function corresponding to Type I is (defined as)

$$Z_{b,I}(x) := H_{\mathbb{H},b,2}((0,x),(\infty,1))$$

and the partition function corresponding to Type II is

$$Z_{b,II}(x) := H_{\mathbb{H},b,2}((0,1-x),(\infty,1)) = Z_{b,I}(1-x).$$

Using our earlier proposition for the multiple SLE partition function and properties of the hypergeometric function:

$$\begin{aligned}\mathbf{P}\{\text{config of Type I}\} &= \frac{Z_{b,I}(x)}{Z_{b,I}(x)+Z_{b,II}(x)} \\ &= \frac{F(2a,6a-1,4a;x)}{F(2a,6a-1,4a;x)+F(2a,6a-1,4a;1-x)}\end{aligned}$$

and

$$\mathbf{P}\{\text{config of Type II}\} = \frac{F(2a,6a-1,4a;1-x)}{F(2a,6a-1,4a;x)+F(2a,6a-1,4a;1-x)}.$$

Summary of results for the 2d critical Ising model

In the case of the Ising model, $\kappa = 3$ (so $b = 1/2$, $a = 2/3$), then the probability of a configuration of Type II is:

$$P_1(x) = \frac{F(\frac{4}{3}, 3, \frac{8}{3}; 1-x)}{F(\frac{4}{3}, 3, \frac{8}{3}; x) + F(\frac{4}{3}, 3, \frac{8}{3}; 1-x)}.$$

Arguin and Saint-Aubin (2002):

$$P_2(x) = \frac{1}{2} - \frac{9}{20}\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})^2}\left[\frac{x^{5/3}(1-x)^{5/3}}{1-x+x^2}\right]\left[F(\tfrac{4}{3}, 3, \tfrac{8}{3}; x) - F(\tfrac{4}{3}, 3, \tfrac{8}{3}; 1-x)\right]$$

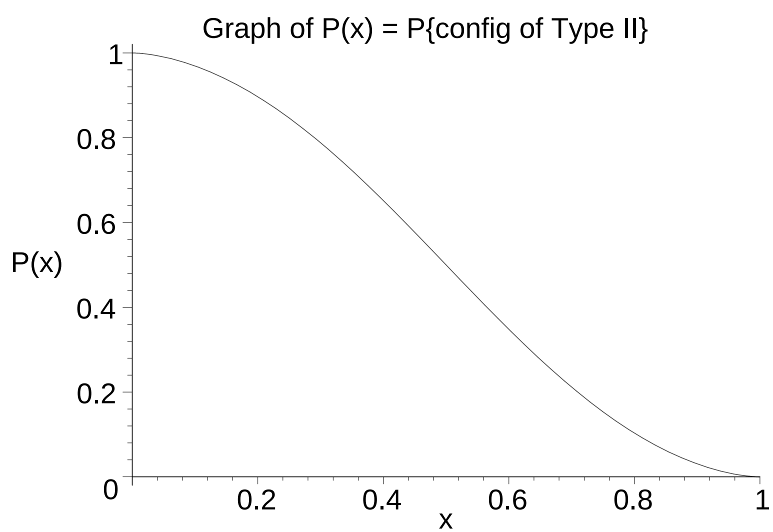
Bauer, Bernard, and Kytölä (2005):

$$P_3(x) = \left(\int_0^1 \frac{y^{2/3}(1-y)^{2/3}}{(1-y+y^2)^2}\,dy\right)^{-1} \int_x^1 \frac{y^{2/3}(1-y)^{2/3}}{(1-y+y^2)^2}\,dy.$$

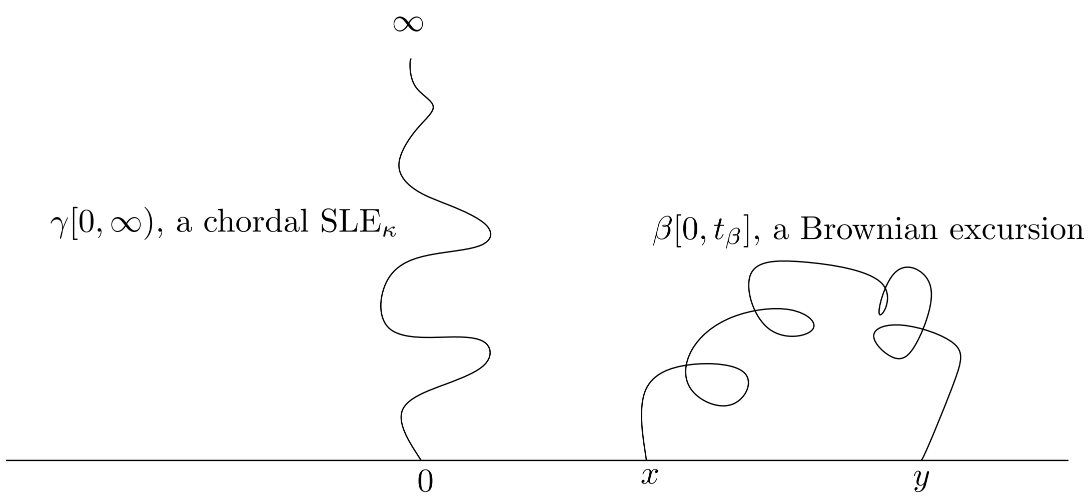
Summary of results for the 2d critical Ising model (cont)

It is not at all obvious that these three expressions are identical.

However, since all three represent the same physical observable (and since each was obtained by solving the same differential equation), it must be the case that $P_1(x) = P_2(x) = P_3(x)$ for $0 \leq x \leq 1$.



An extension to an intersection probability



Theorem. (K. 2009) Suppose that $0 < x < y < \infty$ are real numbers and let $\beta : [0, t_\beta] \to \overline{\mathbb{H}}$ be a Brownian excursion from x to y in $\mathbb{H}$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE_κ, $0 < \kappa \le 4$, from 0 to ∞ in $\mathbb{H}$, then

$$\mathbf{P}\{\, \gamma[0,\infty) \cap \beta[0,t_\beta] = \emptyset \,\} = \frac{\Gamma(2a)\Gamma(4a+1)}{\Gamma(2a+2)\Gamma(4a-1)}\,(x/y)\,F(2, 1-2a, 2a+2; x/y)$$

where $F = {}_2F_1$ is the hypergeometric function and $a = 2/\kappa$.

To Do

SLE describes the scaling limit of a single interface for several models in the $4 < \kappa < 8$ regime such as percolation ($\kappa = 6$) or the FK random cluster model ($\kappa = 16/3$). What about multiple interfaces? Rigorously constructing a measure on multiple non-crossing SLE paths for $4 < \kappa < 8$ is still an open problem.

Other observables? Schramm calculated the probability that a point is to the left of the SLE trace. Extending this to, say, two points to the left of the SLE trace, or two points between the multiple interfaces is still an open problem. Cardy and Simmons (2009) use CFT to give a formula for $\text{SLE}_{8/3}$ (i.e., self-avoiding walk).