The Scaling Limit of Fomin's Identity for Two Paths

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History and References

This talk is based on the short survey paper (which contains the missing details):

The scaling limit of Fomin's identity for two paths, `math.PR/0703615`.

Fomin's original paper established an identity relating (functionals of) loop-erased random walks and (functionals of) simple random walks:

S. Fomin, *Loop-erased walks and total positivity*, Trans Amer Math Soc, '01.

Two papers by MJK and G. Lawler, University of Chicago, followed. The first showed that the functional of SRW converged in the scaling limit to the appropriate functional of Brownian motion.

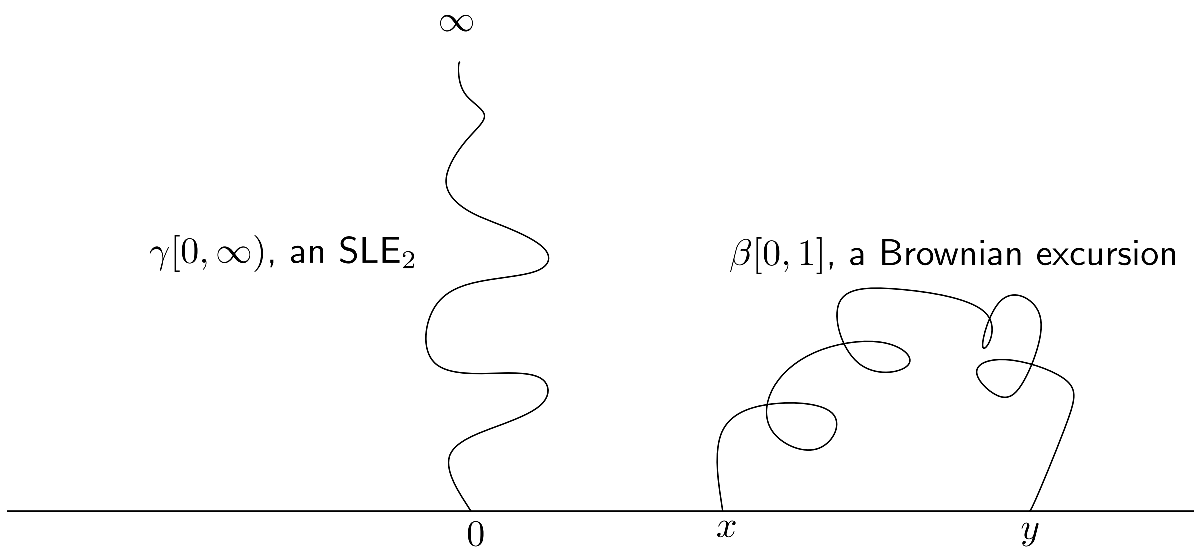
Estimates of random walk exit probabilities and application to LERW, EJP, '05.

The second introduced a configurational measure on mutually avoiding SLE paths ($0 < \kappa \leq 4$) and as a corollary of that work we showed that the functional of LERW converged in the scaling limit to the appropriate functional of SLE_2.

The configurational measure on mutually avoiding SLE paths, Fields Comm 50, '07.

The Main Question

What is the probability that $\gamma[0,\infty)$, a chordal SLE_2 from 0 to ∞ in the upper half plane $\mathbb{H}$, and $\beta[0,1]$, a Brownian excursion from x to y in $\mathbb{H}$, do not intersect (with $0 < x < y < \infty$)?



Question: What is $\mathbf{P}\{\,\gamma[0,\infty) \cap \beta[0,1] = \emptyset\,\}$?

Motivation

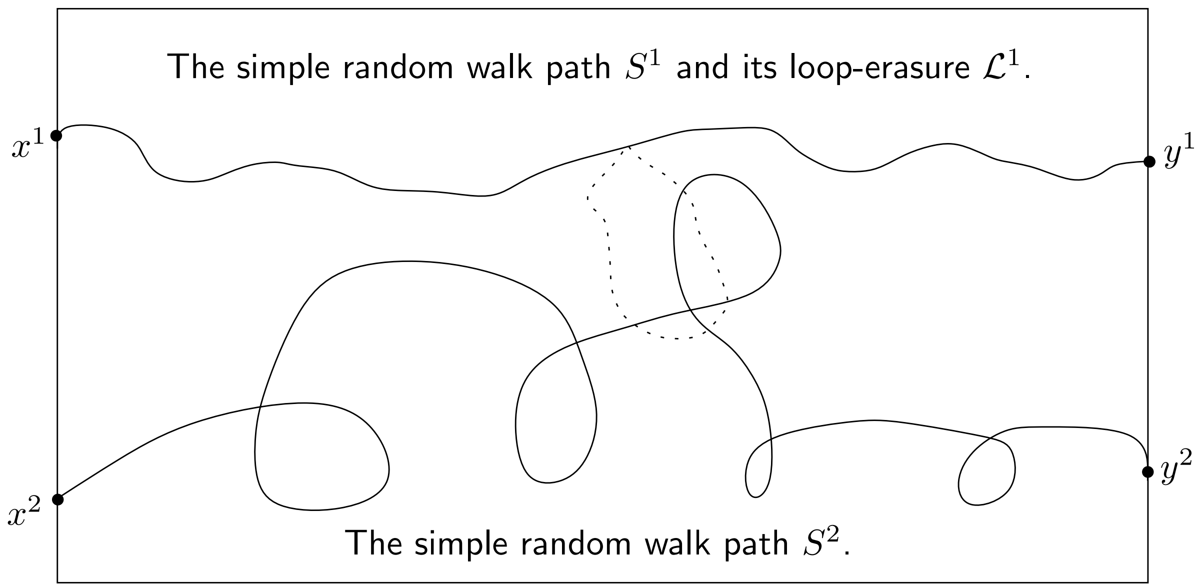
The motivation for asking this question is that the probability under consideration is the natural continuous analogue of the probability that arises in Fomin's identity. In fact, Fomin's original identity expressed the probability of a particular functional of loop-erased random walk in terms of the determinant of the hitting matrix for simple random walk, and in that work he conjectured that this identity holds for continuous processes:

> "... we do not need the notion of loop-erased Brownian motion. Instead, we discretize the model, compute the probability, and then pass to the limit."

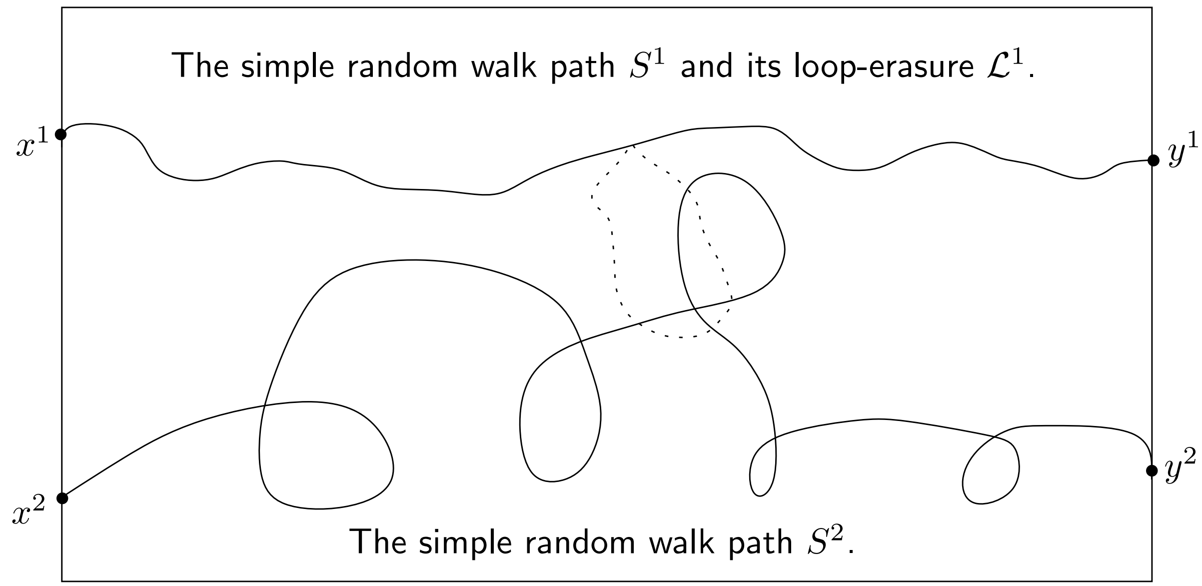
Fomin's identity for two paths

Fomin's original identity actually holds in much more generality and may be viewed as an extension of the Karlin-McGregor formula.

The version we state is for the special case of two simple random walk paths in a finite, simply connected subset $A \subset \mathbb{Z}^2$ connecting pairs of boundary points x^1, x^2, y^2, y^1 ordered counterclockwise around ∂A.



Note: In this example, $\mathcal{L}^1 \cap S^2 = \emptyset$ although $S^1 \cap \mathcal{L}^2 \neq \emptyset$.



Theorem (Fomin): If $\mathcal{L}^1$ is the path of a loop-erased random walk excursion from x^1 to y^1, and S^2 is the path of a simple random walk excursion from x^2 to y^2, then

$$\mathbf{P}\{\, \mathcal{L}^1 \cap S^2 = \emptyset \,\} = \frac{\det \mathbf{h}_{\partial A}(\mathbf{x}, \mathbf{y})}{h_{\partial A}(x^1, y^1)\, h_{\partial A}(x^2, y^2)}$$

$$= \frac{h_{\partial A}(x^1, y^1)\, h_{\partial A}(x^2, y^2) - h_{\partial A}(x^1, y^2)\, h_{\partial A}(x^2, y^1)}{h_{\partial A}(x^1, y^1)\, h_{\partial A}(x^2, y^2)}$$

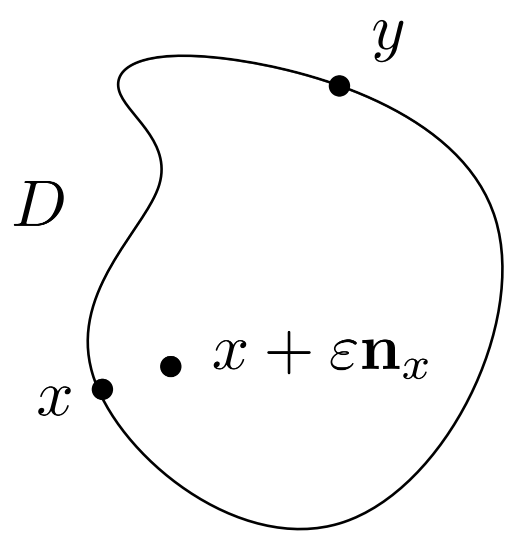
where $h_{\partial A}(x, y) := \mathbf{P}^x\{S_{\tau_A} = y,\ S_1 \in A\}$ is the *discrete excursion Poisson kernel*.

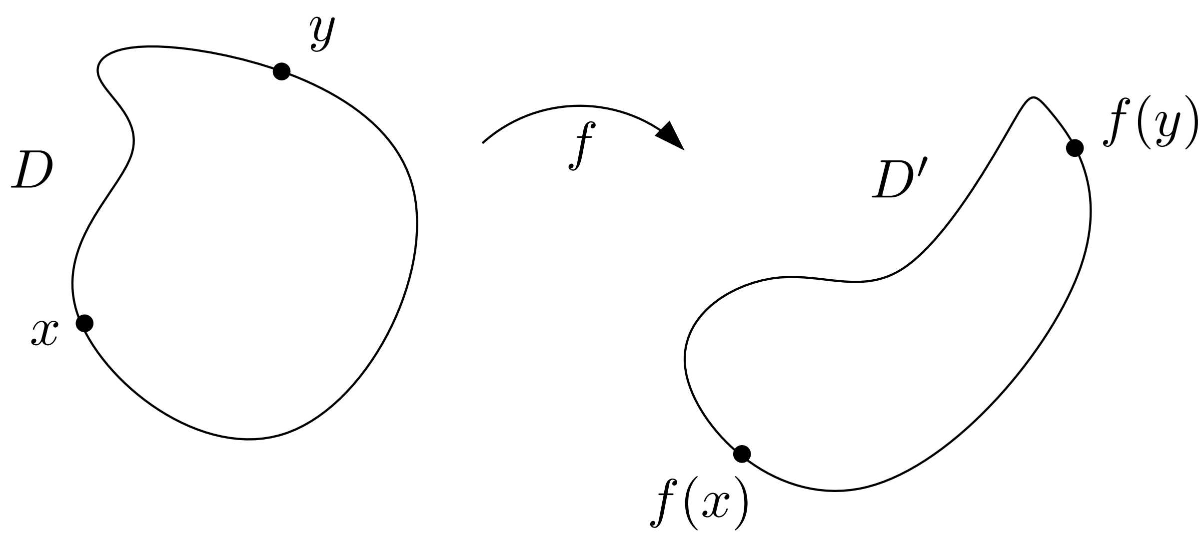
Excursion Poisson Kernel

Suppose that $D \subset \mathbb{C}$ is a simply connected Jordan domain and that ∂D is locally analytic at x and y. The *excursion Poisson kernel* is defined as

$$H_{\partial D}(x,y) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} H_D(x + \varepsilon \mathbf{n}_x, y)$$

where $H_D(z,y)$ for $z \in D$ is the usual Poisson kernel, and $\mathbf{n}_x$ is the unit normal at x pointing into D.





Proposition: If $f : D \to D'$ is a conformal transformation where $D' \subset \mathbb{C}$ is also a simply connected Jordan domain, and $\partial D'$ is locally analytic at $f(x)$, $f(y)$, then

$$H_{\partial D}(x,y) = |f'(x)||f'(y)|H_{\partial D'}(f(x), f(y)).$$

Example: Unit disk $\mathbb{D}$: $H_{\partial \mathbb{D}}(x,y) = \dfrac{1}{\pi\,|y-x|^2} = \dfrac{1}{2\pi(1-\cos(\arg y - \arg x))}$.

Example: Upper half plane $\mathbb{H}$: $H_{\partial \mathbb{H}}(x,y) = \dfrac{1}{\pi\,(y-x)^2}$.

Suppose now that $x^1, \ldots, x^n, y^1, \ldots, y^n$ are distinct boundary points at which ∂D is locally analytic, let $f : D \to D'$ be a conformal transformation, and assume that $\partial D'$ is also locally analytic at $f(x^1), \ldots, f(x^n), f(y^1), \ldots, f(y^n)$. It follows that if $\mathbf{H}_{\partial D}(\mathbf{x}, \mathbf{y}) := [H_{\partial D}(x^i, y^\ell)]_{1 \le i, \ell \le n}$ denotes the $n \times n$ *hitting matrix*

$$\mathbf{H}_{\partial D}(\mathbf{x}, \mathbf{y}) := \begin{bmatrix} H_{\partial D}(x^1, y^1) & \cdots & H_{\partial D}(x^1, y^n) \\ \vdots & \ddots & \vdots \\ H_{\partial D}(x^n, y^1) & \cdots & H_{\partial D}(x^n, y^n) \end{bmatrix}$$

then conformal covariance implies

$$\det \mathbf{H}_{\partial D}(\mathbf{x}, \mathbf{y}) = \left(\prod_{j=1}^{n} |f'(x^j)| \, |f'(y^j)| \right) \det[H_{\partial D'}(f(x^i), f(y^\ell))]_{1 \le i, \ell \le n}. \quad (\dagger)$$

It now follows from (†) that

$$\frac{\det \mathbf{H}_{\partial D}(\mathbf{x}, \mathbf{y})}{\displaystyle\prod_{i=1}^{n} H_{\partial D}(x^i, y^i)}$$

is a conformal *invariant*.

The non-intersection probability of SLE$_2$ and Brownian motion

Theorem: If $0 < x < y < \infty$ are real numbers, $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_2$ from 0 to ∞ in $\mathbb{H}$, and $\beta : [0, 1] \to \overline{\mathbb{H}}$ is a Brownian excursion from x to y in $\mathbb{H}$, then

$$\mathbf{P}\{\, \gamma[0, \infty) \cap \beta[0, 1] = \emptyset \,\} = \frac{\det \mathbf{H}_{\partial\mathbb{D}}(f(\mathbf{x}), f(\mathbf{y}))}{H_{\partial\mathbb{D}}(f(0), f(\infty))\, H_{\partial\mathbb{D}}(f(x), f(y))} \qquad (*)$$

where $f : \mathbb{H} \to \mathbb{D}$ is a conformal transformation.

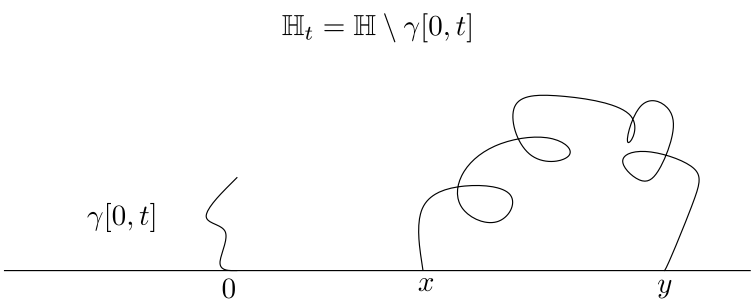
Strategy for the Proof

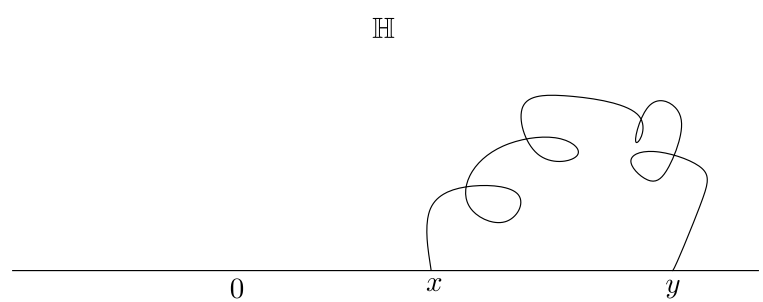
Our strategy for establishing this result will be as follows. We will first determine an explicit expression for $\mathbf{P}\{\, \gamma[0, \infty) \cap \beta[0, 1] = \emptyset \,\}$, and we will then show that this explicit expression is the same as the right side of $(*)$.

Proof

For $0 < t < \infty$, let $\mathbb{H}_t$ denote the slit-plane $\mathbb{H}_t = \mathbb{H} \setminus \gamma(0, t]$ so that

$$\mathbf{P}\{\gamma[0,t] \cap \beta[0,1] = \emptyset\} = \mathbb{E}^{x,y}\left[\frac{H_{\partial\mathbb{H}_t}(x,y)}{H_{\partial\mathbb{H}}(x,y)}\right].$$





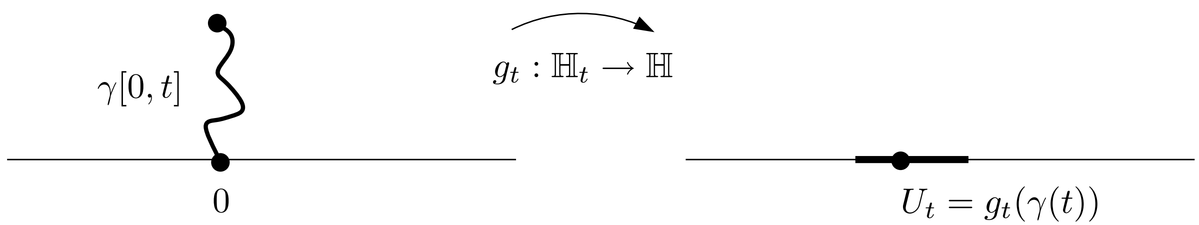
Letting $t \to \infty$ implies

$$\mathbf{P}\{\gamma[0,\infty) \cap \beta[0,1] = \emptyset\} = \mathbb{E}^{x,y}\left[\lim_{t\to\infty} \frac{H_{\partial\mathbb{H}_t}(x,y)}{H_{\partial\mathbb{H}}(x,y)}\right].$$

Let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the unique conformal transformation satisfying the hydrodynamic normalization $g_t(z) - z = o(1)$ as $z \to \infty$. It is well-known that g_t satisfies the chordal Loewner equation, namely

$$\frac{\partial}{\partial t} g_t(z) = \frac{1}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = -B_t$ is a standard Brownian motion.



We now map $\mathbb{H}_t$ to $\mathbb{H}$ by g_t and use conformal covariance to conclude that

$$H_{\partial\mathbb{H}_t}(x,y) = g_t'(x) g_t'(y) H_{\partial\mathbb{H}}(g_t(x), g_t(y))$$

and so

$$\frac{H_{\partial\mathbb{H}_t}(x,y)}{H_{\partial\mathbb{H}}(x,y)} = \frac{g_t'(x) g_t'(y) H_{\partial\mathbb{H}}(g_t(x), g_t(y))}{H_{\partial\mathbb{H}}(x,y)} = (y-x)^2 \cdot \frac{g_t'(x) g_t'(y)}{(g_t(y) - g_t(x))^2}$$

where the last equality follows from the explicit form of $H_{\partial\mathbb{H}}$.

Recall: $\dfrac{H_{\partial \mathbb{H}_t}(x,y)}{H_{\partial \mathbb{H}}(x,y)} = (y-x)^2 \cdot \dfrac{g_t'(x)g_t'(y)}{(g_t(y)-g_t(x))^2}$

Let

$$J_t := \frac{g_t'(x)g_t'(y)}{(g_t(y)-g_t(x))^2} \quad \text{and set} \quad J_\infty := \lim_{t\to\infty} J_t.$$

Let $P(x,y) := \mathbf{P}\{\gamma[0,\infty) \cap \beta[0,1] = \emptyset\}$ so that

$$P(x,y) = (y-x)^2 \mathbb{E}^{x,y}\left[\lim_{t\to\infty} \frac{g_t'(x)g_t'(y)}{(g_t(y)-g_t(x))^2}\right] = (y-x)^2 \mathbb{E}^{x,y}[J_\infty].$$

In order to determine $P(x,y)$, we will derive and solve an ODE for it.

Let $X_t := g_t(x) + B_t$ and $Y_t := g_t(y) + B_t$ so that

$$dX_t = \frac{1}{X_t} dt + dB_t \quad \text{and} \quad dY_t = \frac{1}{Y_t} dt + dB_t.$$

Some routine calculations give

$$\frac{\partial}{\partial t} \log g_t'(x) = -\frac{1}{X_t^2}, \; \frac{\partial}{\partial t} \log g_t'(y) = -\frac{1}{Y_t^2}, \text{ and } \frac{\partial}{\partial t} \log(g_t(y) - g_t(x)) = -\frac{1}{X_t\, Y_t},$$

and so we see that

$$J_t = J_0 \, \exp\left\{ \int_0^t \partial_s [\log J_s] \, ds \right\} = \frac{1}{(y-x)^2} \, \exp\left\{ -\int_0^t \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 \, ds \right\}.$$

Hence, putting things together we find

$$P(x,y) = \mathbb{E}^{x,y} \left[\exp\left\{ -\int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 \, ds \right\} \right].$$

It now follows from the usual Markov property that $J_t\, P(X_t, Y_t)$ is a martingale. That is, if $M_t := \mathbb{E}^{x,y}[J_\infty | \mathcal{F}_t]$ so that M_t is a martingale, then

$$
\begin{aligned}
M_t &= \mathbb{E}^{x,y}\left[\frac{1}{(y-x)^2}\ \exp\left\{-\int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s}\right)^2\, ds\right\}\ \Big|\ \mathcal{F}_t\right] \\
&= \frac{1}{(y-x)^2} \exp\left\{-\int_0^t \left(\frac{1}{X_s} - \frac{1}{Y_s}\right)^2\, ds\right\} \cdot \\
&\qquad\qquad \cdot\, \mathbb{E}^{x,y}\left[\exp\left\{-\int_t^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s}\right)^2\, ds\right\}\ \Big|\ \mathcal{F}_t\right] \\
&= J_t\, P(X_t, Y_t).
\end{aligned}
$$

Itô's formula now implies that

$$
-\left(\frac{1}{x} - \frac{1}{y}\right)^2 P + \frac{1}{x}\,\frac{\partial P}{\partial x} + \frac{1}{y}\,\frac{\partial P}{\partial y} + \frac{1}{2}\,\frac{\partial^2 P}{\partial x^2} + \frac{1}{2}\,\frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial x \partial y} = 0.
$$

Since the probability in question only depends on the ratio x/y, we see that $P(x,y) = \varphi(x/y)$ for some function φ. Thus, letting $u = x/y$ and noting that $0 < u < 1$, we find

$$
u^2\,(1-u)\,\varphi''(u) + 2\,u\,\varphi'(u) - 2(1-u)\,\varphi(u) = 0.
$$

The constraint $0 < u < 1$ allows us to consider $\psi(u) := u^{-1}(1-u)^{-3}\varphi(u)$ which satisfies the ODE

$$u\,(1-u)\,\psi''(u) + (4-8u)\,\psi'(u) - 10\,\psi(u) = 0.$$

This is the well-known hypergeometric differential equation, and so

$$\psi(u) = C_1\,\frac{2-u}{(1-u)^3} + C_2\,\frac{1-2u}{u^3(1-u)^3}$$

which implies that

$$\varphi(u) = C_1 u(2-u) + C_2 u^{-2}(1-2u).$$

However, physical considerations dictate that $\varphi(u) \to 0$ as $u \to 0+$ and $\varphi(u) \to 1$ as $u \to 1-$, and so $C_2 = 0$ and $C_1 = 1$.

Thus, $\varphi(u) = u(2-u)$ and so we find

$$\mathbf{P}\{\gamma[0,\infty) \cap \beta[0,1] = \emptyset\} = P(x,y) = \varphi(x/y) = \frac{x}{y}\left(2 - \frac{x}{y}\right).$$

As already noted, the probability in question only depends on the ratio x/y, and so it suffices without loss of generality to assume that $0 < x < 1$ and $y = 1$.

Furthermore, we may assume that the conformal transformation $f : \mathbb{H} \to \mathbb{D}$ is

$$f(z) = \frac{iz+1}{z+i},$$

so that $f(0) = -i$, $f(y) = f(1) = 1$, $f(\infty) = i$, and

$$f(x) = \left(\frac{2x}{x^2+1}\right) + i\left(\frac{x^2-1}{x^2+1}\right) = \exp\left\{-i\arctan\left(\frac{1-x^2}{2x}\right)\right\}.$$

Writing $f(x) = e^{i\theta}$, we find that

$$\begin{aligned}
&\frac{\det \mathbf{H}_{\partial\mathbb{D}}(f(\mathbf{x}), f(\mathbf{y}))}{H_{\partial\mathbb{D}}(f(0), f(\infty))\, H_{\partial\mathbb{D}}(f(x), f(y))} \\
&\qquad = \frac{H_{\partial\mathbb{D}}(-i,i)\, H_{\partial\mathbb{D}}(e^{i\theta},1) - H_{\partial\mathbb{D}}(-i,1)\, H_{\partial\mathbb{D}}(e^{i\theta},i)}{H_{\partial\mathbb{D}}(-i,i)\, H_{\partial\mathbb{D}}(e^{i\theta},1)} \\
&\qquad = \frac{2\cos\theta + \sin\theta - 1}{1+\sin\theta}.
\end{aligned}$$

Since $\theta = -\arctan\left(\frac{1-x^2}{2x}\right)$ we see that $\cos\theta = \frac{2x}{x^2+1}$ and $\sin\theta = \frac{1-x^2}{x^2+1}$ which upon substitution gives

$$\frac{2\cos\theta + \sin\theta - 1}{1+\sin\theta} = \frac{\frac{4x}{x^2+1} + \frac{1-x^2}{x^2+1} - 1}{1 + \frac{1-x^2}{x^2+1}} = \frac{4x - 2x^2}{2} = x(2-x).$$

Comparing this with our earlier result proves the theorem. □

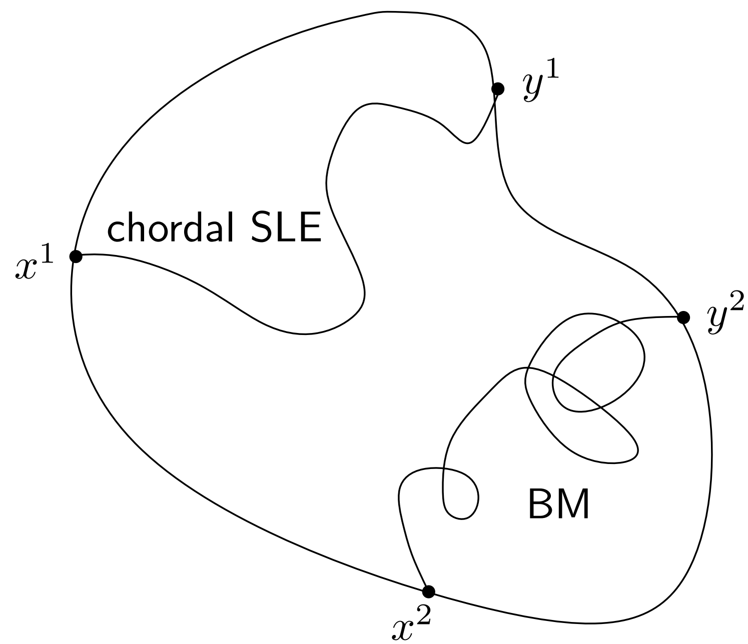
An Example

Example: Let $x = 1/2$ and $y = 1$. Then $f : \mathbb{H} \to \mathbb{D}$ has $f(0) = i$, $f(1) = 1$, $f(\infty) = -i$, and $f(1/2) = \exp\{-i \arctan(3/4)\}$. A simple calculation gives

$$\mathbf{P}\{\gamma[0,\infty) \cap \beta[0,1] = \emptyset\} = \frac{2 \cdot \frac{4}{5} + \frac{3}{5} - 1}{1 + \frac{3}{5}} = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}.$$

Corollary

Suppose that $D \subset \mathbb{C}$ is a bounded, simply connected planar domain, and that x^1, x^2, y^2, y^1 are four points ordered counterclockwise around ∂D. The probability a chordal SLE_2 from x^1 to y^1 in D does not intersect a Brownian excursion from x^2 to y^2 in D is $\Phi(x^2)\left(2 - \Phi(x^2)\right)$ where $\Phi : D \to \mathbb{H}$ is the conformal transformation with $\Phi(x^1) = 0$, $\Phi(y^1) = \infty$, $\Phi(y^2) = 1$.



This statement can be easily modified to cover the case when D is unbounded and/or the case when ∞ is one of the boundary points.