Notes from *Stirling's Formula: An Application of Calculus*

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Recall that for $N \in \mathbb{N}$, $\Gamma(N) = \int_0^\infty e^{-t}\, t^{N-1}\, dt$ which can be extended to any $x \in \mathbb{R}^+$ as

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt.$$

Laplace's method tells us that for appropriate f

$$\int_{-\infty}^{\infty} e^{Nf(x)}\, dx \simeq \frac{\sqrt{2\pi}\, e^{Nf(x_0)}}{\sqrt{-Nf''(x_0)}}.$$

Stirling's formula says

$$\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi}\, e^{-N}\, N^{N+\frac{1}{2}}} = 1.$$

Online Extensions

There are many different proofs of Stirling's Formula. Others which also requires only first-year calculus may be found at:

http://www.sosmath.com/calculus/sequence/stirling/stirling.html
http://140.122.140.53/∼yclin/02a/cx/cx22.pdf

For an interesting discussion about extended Stirling Formulas, and for those with a background in computer science, an interesting discussion of numerically approximating the Gamma function, check out: http://www.rskey.org/gamma.htm

Homework!

1. Check over the double integral calculation $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{(-x^2-y^2)/2}\, dx\, dy = 2\pi$. Don't forget about the Jacobian for polar coordinates.
2. Check the computation that $N! = \Gamma(N+1)$.
3. Carefully show that $\int_0^\infty e^{-t}\, t^{x-1}\, dt$ converges for $0 < x < \infty$.
4. By changing variables, show $\Gamma(x+1) = x\,\Gamma(x)$.
5. By changing variables, show $\Gamma(1/2) = \int_{-\infty}^{\infty} e^{-x^2}\, dx$.
6. Demonstrate that $\Gamma(1/2) = \sqrt{\pi}$ is equivalent to $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$.
7. Check using Laplace's method that $\int_0^\pi x^N \sin x\, dx \simeq \pi^{N+2} N^{-2}$.
8. Check using Stirling's formula that for even N,

$$\frac{N!}{(N/2)!} \simeq 2^N \sqrt{\frac{2}{\pi N}}.$$

(This is basically the deMoivre-Laplace local central limit theorem.)