The configurational measure on mutually avoiding SLE paths

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–*Estimates of random walk exit probabilities and application to LERW*, Elect. J. Probab., 2005.

–*The configurational measure on mutually avoiding SLE paths*, `arXiv:math.PR/0605159`.

The Basic Setup

– $D \subset \mathbb{C}$ simply connected, ∂D Jordan

– $z_1, \ldots, z_n, w_n, \ldots, w_1$ distinct points ordered counterclockwise on ∂D

– write $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (w_1, \ldots, w_n)$

– fix a parameter $b \in \mathbb{R}$ (boundary scaling exponent or boundary conformal weight)

Goal: To define a measure

$$Q_{D,b,n}(\mathbf{z}, \mathbf{w})$$

on mutually avoiding n-tuples $(\gamma^1, \ldots, \gamma^n)$ of simple paths in D, and satisfying certain properties:

(1) conformal covariance (2) boundary perturbation

(3) cascade relation (4) Markov property

Note that $\gamma^i : [0,1] \to \mathbb{C}$ with $\gamma^i(0) = z_i$, $\gamma^i(1) = w_i$, $\gamma(0,1) \subset D$.

Conformal Covariance

If D is analytic at $\mathbf{z}$, $\mathbf{w}$, then $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ is a non-zero, finite measure supported on n-tuples $(\gamma^1,\ldots,\gamma^n)$ where γ^j is a simple curve in D connecting z_j and w_j and

$$\gamma^j \cap \gamma^k = \emptyset, \quad 1 \leq j < k \leq n.$$

Moreover, if $f : D \to f(D)$ is a conformal transformation and $f(D)$ is analytic at $f(\mathbf{z})$, $f(\mathbf{w})$, then

$$f \circ Q_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b \, |f'(\mathbf{w})|^b \, Q_{f(D),b,n}(f(\mathbf{z}), f(\mathbf{w})) \tag{1}$$

where $f(\mathbf{z}) = (f(z_1),\ldots,f(z_n))$ and $f'(\mathbf{z}) = f'(z_1)\cdots f'(z_n)$.

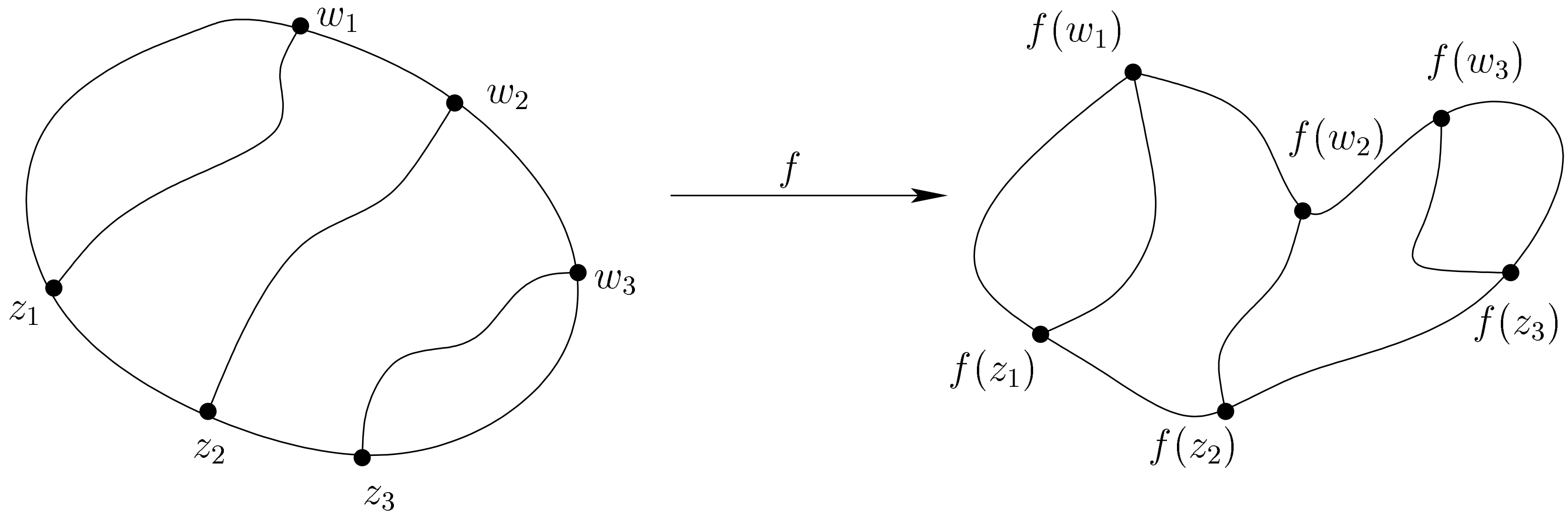


Figure 1: Conformal Covariance

Recall: $f \circ Q_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b\,|f'(\mathbf{w})|^b\,Q_{f(D),b,n}(f(\mathbf{z}),f(\mathbf{w}))$ (1)

Write

$$Q_{D,b,n}(\mathbf{z},\mathbf{w}) = H_{D,b,n}(\mathbf{z},\mathbf{w})\,\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w}),$$

where $H_{D,b,n}(\mathbf{z},\mathbf{w}) = |Q_{D,b,n}(\mathbf{z},\mathbf{w})|$ and $\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w})$ is a probability measure.

The conformal covariance condition (1) then becomes the scaling rule for H,

$$H_{D,b,n}(\mathbf{z},\mathbf{w}) = |f'(\mathbf{z})|^b\,|f'(\mathbf{w})|^b\,H_{f(D),b,n}(f(\mathbf{z}),f(\mathbf{w})), \tag{2}$$

and the conformal *invariance* rule for $\mu^{\#}$,

$$f \circ \mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w}) = \mu^{\#}_{f(D),b,n}(f(\mathbf{z}),f(\mathbf{w})). \tag{3}$$

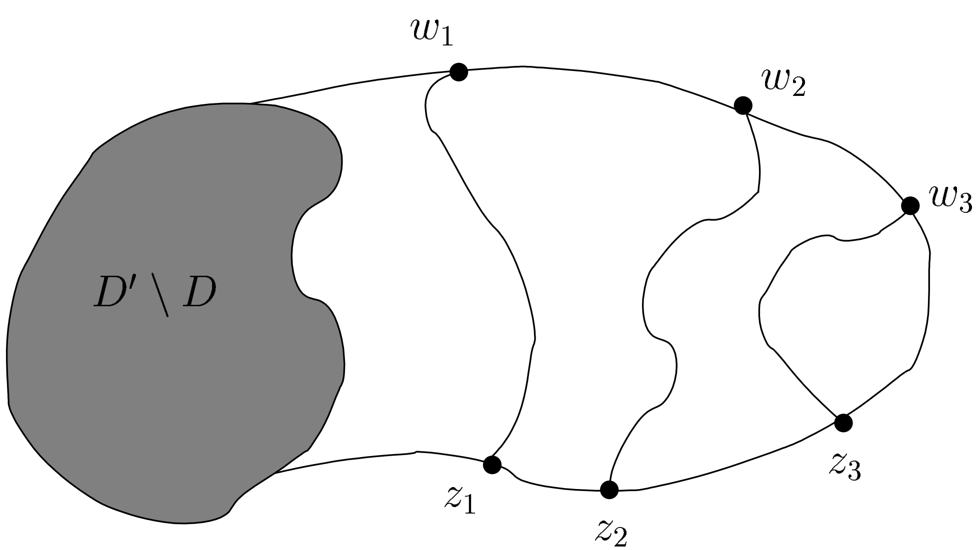
Since $\mu^{\#}$ is a conformal invariant, we can define $\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w})$ even if the boundaries are not smooth at $\mathbf{z}$, $\mathbf{w}$.

Boundary Perturbation

Suppose $D \subset D'$ are Jordan domains and ∂D, $\partial D'$ agree and are analytic in neighbourhoods of $\mathbf{z}$, $\mathbf{w}$. Then $Q_{D,b,n}(\mathbf{z},\mathbf{w})$ is absolutely continuous with respect to $Q_{D',b,n}(\mathbf{z},\mathbf{w})$. Moreover, the Radon-Nikodym derivative

$$Y_{D,D',b,n}(\mathbf{z},\mathbf{w}) = \frac{dQ_{D,b,n}(\mathbf{z},\mathbf{w})}{dQ_{D',b,n}(\mathbf{z},\mathbf{w})}$$

is a conformal invariant.



Recall: $D \subset D'$ and

$$Y_{D,D',b,n}(\mathbf{z},\mathbf{w}) = \frac{dQ_{D,b,n}(\mathbf{z},\mathbf{w})}{dQ_{D',b,n}(\mathbf{z},\mathbf{w})}$$

Saying that $Y_{D,D',b,n}(\mathbf{z},\mathbf{w})$ is a conformal invariant means that if $f : D' \to f(D')$ is a conformal map that extends analytically in neighbourhoods of $\mathbf{z}$, $\mathbf{w}$, then

$$Y_{f(D),f(D'),b,n}(f(\mathbf{z}), f(\mathbf{w}))(f \circ \bar{\gamma}) = Y_{D,D',b,n}(\mathbf{z},\mathbf{w})(\bar{\gamma}), \tag{4}$$

where $\bar{\gamma} = (\gamma^1, \ldots, \gamma^n)$ and $f \circ \bar{\gamma} = (f \circ \gamma^1, \ldots, f \circ \gamma^n)$.

As with $\mu^{\#}_{D,b,n}(\mathbf{z},\mathbf{w})$, the last condition (4) implies that $Y_{D,D',b,n}(\mathbf{z},\mathbf{w})$ is well-defined even if the boundaries are not smooth at $\mathbf{z}, \mathbf{w}$.

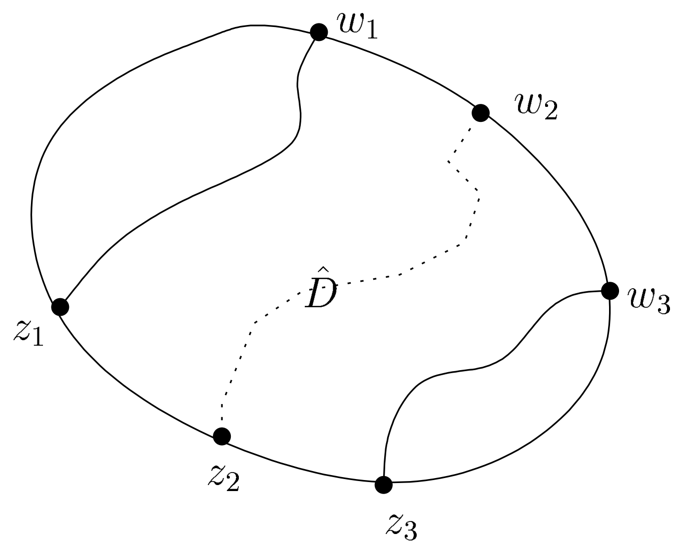
Cascade Relation

Let

$$\hat{\mathbf{z}} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n), \quad \hat{\mathbf{w}} = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n),$$

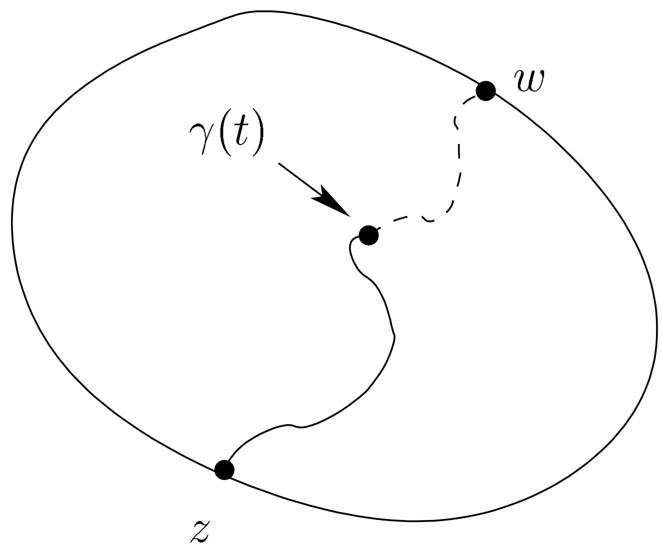
$$\hat{\boldsymbol{\gamma}} = (\gamma^1, \ldots, \gamma^{j-1}, \gamma^{j+1}, \ldots, \gamma^n).$$

The marginal distribution on $\hat{\boldsymbol{\gamma}}$ induced by $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ is absolutely continuous with respect to $Q_{D,b,n-1}(\hat{\mathbf{z}}, \hat{\mathbf{w}})$ with Radon-Nikodym derivative $H_{\hat{D},b,1}(z_j, w_j)$. Here $\hat{D}$ is the subdomain of $D \setminus \hat{\boldsymbol{\gamma}}$ whose boundary includes z_j, w_j.



Markov Property

In the measure $\mu^{\#}_{D,b,1}(z,w)$, the conditional distribution on γ given an initial segment $\gamma[0,t]$ is $\mu^{\#}_{D\setminus\gamma[0,t],b,1}(\gamma(t),w)$.



Note: We have stated this condition in a way that does not use two dimensions and conformal invariance.

Schramm's Result

Note: The *conformal Markov property* is the combination of the Markov property and (3). O. Schramm showed that there is a one-parameter family of measures, which he parametrized by κ, satisfying the conformal Markov property. While these measures are well-defined for $\kappa > 0$, they are supported on simple curves only for $0 < \kappa \leq 4$.

Existence of the Configurational Measure

Theorem (Kozdron-Lawler): *For any $b \geq \frac{1}{4}$, there exists a family of measures $Q_{D,b,n}(\mathbf{z}, \mathbf{w})$ supported on n-tuples of mutually avoiding simple curves satisfying*

- *conformal covariance*
- *boundary perturbation*
- *cascade relation*
- *Markov property*

Moreover, the simple curve γ^i is a chordal SLE$_\kappa$ from z_i to w_i in D where

$$\kappa = \frac{6}{2b+1}.$$

Note: $b \geq \frac{1}{4} \longleftrightarrow 0 < \kappa \leq 4$

The Partition Function for Two Paths

By conformal invariance, it suffices to work in $D = \mathbb{H}$.

If $0 < x_1 < \cdots < x_n < y_n < \cdots < y_1 < \infty$, let

$$H^*_{\mathbb{H},b,n}(\mathbf{x},\mathbf{y}) = \lim_{w\to\infty} w^{2b}\, H_{\mathbb{H},b,n+1}((0,\mathbf{x}),(w,\mathbf{y})).$$

Proposition: If $b \geq 1/4$ and $n+1 = 2$, then

$$H^*_{\mathbb{H},b,1}(x,y) = (y-x)^{-2b}\, \frac{\Gamma(2a)\,\Gamma(6a-1)}{\Gamma(4a)\,\Gamma(4a-1)}\,(x/y)^a\, F(2a, 1-2a, 4a; x/y)$$

where F denotes the hypergeometric function and $a = \dfrac{2}{\kappa} = \dfrac{2b+1}{3}$.

Note: This result first appeared in J. Dubédat, and was derived non-rigorously by M. Bauer, D. Bernard, and K. Kytölä. Our configurational approach provides another rigorous derivation.

The Scaling Limit of Fomin's Identity

Theorem (Kozdron-Lawler): *If* $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ *is an* SLE_2 *in the upper half plane* $\mathbb{H}$ *from* 0 *to* ∞, *and* $\beta : [0, 1] \to \overline{\mathbb{H}}$ *is a Brownian excursion from* x *to* y *in* $\mathbb{H}$ *where* $0 < x < y < \infty$, *then*

$$\mathbf{P}\{\,\gamma[0,\infty) \cap \beta[0,1] = \emptyset\,\} = 1 - \frac{H\left(f(0), f(y)\right)\ H\left(f(x), f(\infty)\right)}{H\left(f(0), f(\infty)\right)\ H\left(f(x), f(y)\right)}$$

where $f : \mathbb{H} \to \mathbb{D}$ *is a conformal transformation of the upper half plane* $\mathbb{H}$ *onto the unit disk* $\mathbb{D}$, *and* $H(z, w)$ *is the excursion Poisson kernel in* $\mathbb{D}$ *given by*

$$H(z,w) := H_{\partial \mathbb{D}}(z,w) := \frac{1}{\pi}\,\frac{1}{|w-z|^2} = \frac{1}{2\pi}\,\frac{1}{1 - \cos(\arg w - \arg z)}.$$

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