An Introduction to the Loewner Equation and SLE

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Charles Loewner

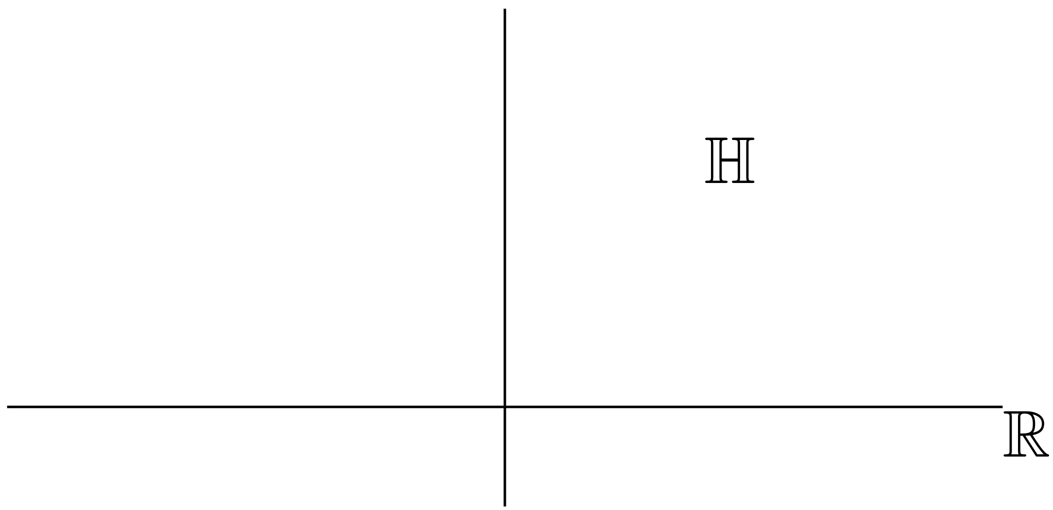
- 1893: born May 29 as Karl Löwner in Lany, Bohemia
- 1917: Ph.D. from University of Prague in geometric function theory under Georg Pick
- 1933: jailed during Nazi occupation of Prague, emmigrated to US, changed his name to Charles Loewner, and received Assistant Professorship at Louisville University
- Brown University (1944-1946); Syracuse University (1946-1951); Stanford University (1951-1968)

Brief History of the Loewner Equation

- (Loewner 1923): proved a special case of the Bieberbach conjecture ($|a_3| \leq 3$)
- (DeBranges 1985): proved entire Bieberbach conjecture
- (Schramm 1999): scaling limits of certain stochastic processes
- (Lawler, Schramm, Werner 2000): proved Mandelbrot's conjecture that dimension of Brownian frontier is 4/3

Mapping $\mathbb{H}$ to $\mathbb{H}$

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane.



Let $h : \mathbb{H} \to \mathbb{H}$ be onto with $h(\infty) = \infty$.

Then h must be of the form $h(z) = az + b$ where $a > 0$ and $b \in \mathbb{R}$.

Riemann Mapping Theorem

The Riemann mapping theorem states that any simply connected proper subset of the complex plane can be mapped conformally onto the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem (Riemann)**.** *Let D be a simply connected domain which is a proper subset of complex plane. Let $z_0 \in D$ be a given point. Then there exists a unique analytic function g which maps D conformally onto $\mathbb{D}$ and has the properties $g(z_0) = 0$ and $g'(z_0) > 0$.*

Mapping $\mathbb{H} \setminus K$ to $\mathbb{H}$

Suppose K is a bounded, compact set such that $\mathbb{H} \setminus K$ is simply connected.

By the Riemann mapping theorem there exist many conformal maps g_K from $\mathbb{H} \setminus K$ to $\mathbb{H}$ with $g_K(\infty) = \infty$.

Using the Schwarz reflection principle, as $z \to \infty$ we can expand g_K around ∞.

$$\therefore \; g_K(z) = bz + a_0 + \frac{a_1}{z} + O\left(\frac{1}{z^2}\right)$$

with $b > 0$ and $a_i \in \mathbb{R}$.

Consider the expansion of $f(z) = [g_K(1/z)]^{-1}$ about the origin. f locally maps $\mathbb{R}$ to $\mathbb{R}$ so the coefficients in the expansion are real and $b > 0$.

For convenience, we choose the unique g_K which satisfies the "hydrodynamic normalization"

$$\lim_{z \to \infty} (g_K(z) - z) = 0.$$

i.e. we choose $b = 1$, $a_0 = 0$

The constant $a(K) := a_1$ only depends on the set K.

Thus $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ with $g_K(\infty) = \infty$ is

$$g_K(z) = z + \frac{a(K)}{z} + O\left(\frac{1}{z^2}\right).$$

Slit Mappings

Let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be a simple curve (no self intersections) with $\gamma(0) = 0$ and $\gamma(0, \infty) \subseteq \mathbb{H}$.

For each $t \geq 0$ suppose that $K_t := \gamma[0, t]$.

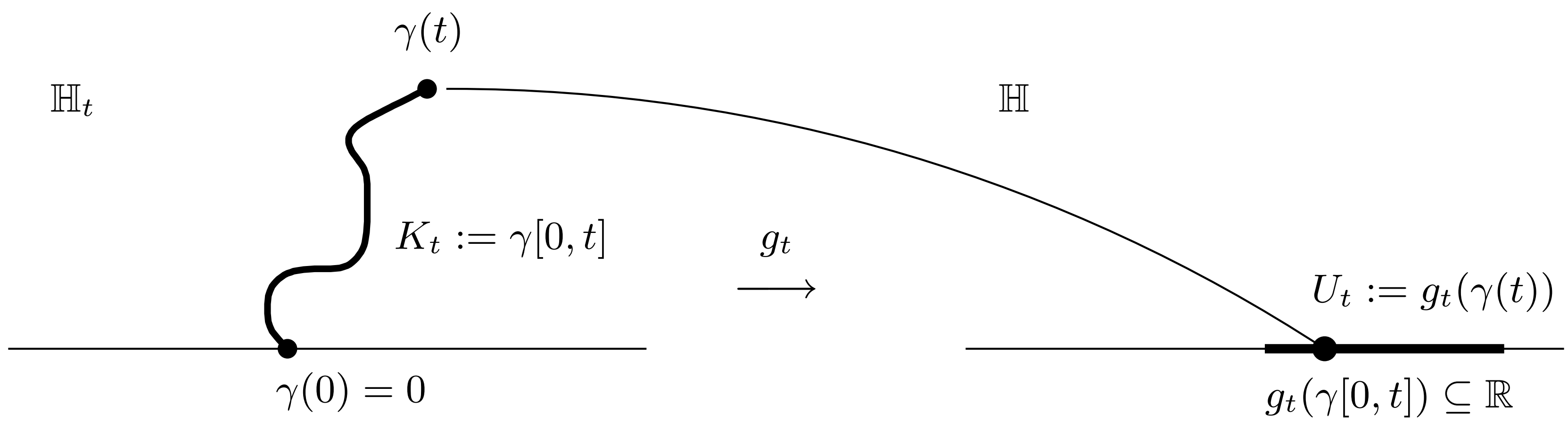
Let $\mathbb{H}_t := \mathbb{H} \setminus K_t$ be the slit half plane and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding Riemann map.

We want $g_t(\infty) = \infty$ and g_t to satisfy hydrodynamic normalization.

Thus as $z \to \infty$,

$$g_t(z) = z + \frac{a(t)}{z} + O\left(\frac{1}{z^2}\right)$$

where $a(t) := a(K_t)$.



The slit half plane $\mathbb{H}_t$ and the corresponding Riemann map to $\mathbb{H}$.

- The curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ evolves from $\gamma(0) = 0$ to $\gamma(t)$.
- $K_t := \gamma[0, t]$, $\mathbb{H}_t := \mathbb{H} \setminus K_t$, $g_t : \mathbb{H}_t \to \mathbb{H}$
- $U_t := g_t(\gamma(t))$, the image of $\gamma(t)$.
- By the Carathéodory extension theorem, $g_t(\gamma[0, t]) \subseteq \mathbb{R}$.

Understanding $a(t)$

As before $K_t = \gamma[0,t]$, $\mathbb{H}_t = \mathbb{H} \setminus K_t$, and

$$a(t) := a(K_t) = a(\gamma[0,t]).$$

We choose the parametrization of $\gamma(t)$ such that $a(t) = 2t$.

i.e. Let $\sigma_t = \inf\{s : a(\gamma(s)) = 2t\}$. Then σ_t is such that $a(\gamma[0,\sigma_t]) = 2t$. Reparametrize by $\tilde{\gamma}(t) = \gamma(\sigma_t)$. Just call this γ.

Facts.

1) if $s < t$, then $a(s) < a(t)$

2) $s \mapsto a(s)$ is continuous

3) $a(0) = 0$, $a(t) \to \infty$ as $t \to \infty$

The Loewner Equation

Assume that $\gamma(t)$ is chosen so that $a(t) = 2t$.

Suppose $K_t := \gamma[0,t]$ with $\mathbb{H}_t := \mathbb{H} \setminus K_t$ and let $g_t : \mathbb{H}_t \to \mathbb{H}$ be the corresponding maps. Let $U_t := g_t(\gamma(t))$.

Then g_t satisfies the Loewner differential equation with the identity map as initial data.

Theorem (Loewner). *$g_t(z)$, for fixed z, is the solution of the IVP*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

The natural thing to do is to start with U_t and solve the Loewner equation.

Suppose that the function $t \mapsto U_t$, $t \in [0, \infty)$ is continuous and real-valued.

Solving the Loewner equation gives g_t which conformally map $\mathbb{H}_t$ to $\mathbb{H}$ where $\mathbb{H}_t = \{z : g_t(z) \text{ is well-defined}\} = \mathbb{H} \setminus K_t$.

Ideally, we would like $g_t^{-1}(U_t)$ to be a well-defined curve so that we can define $\gamma(t) = g_t^{-1}(U_t)$. Although for many choices of U this is not possible, the following theorem gives a sufficient condition.

Theorem (Rohde-Marshall)**.** *If U is "nice" [Hölder 1/2 continuous with sufficiently small Hölder 1/2 norm], then $\gamma(t) = g_t^{-1}(U_t)$ is a well-defined simple curve and $K_t = \gamma[0, t]$.*

Brownian Motion

Brownian motion is a model of "continuous, random motion." Think of Brownian motion as the limit of a random walk where the step sizes get smaller and smaller (and the grid gets finer and finer).

Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ for all i.

If $S_n = X_1 + X_2 + \cdots + X_n$, then $\dfrac{S_n}{\sqrt{n}} \rightarrow$BM (in distribution as $n \rightarrow \infty$).

One dimensional Brownian motion is a real-valued process on the line; $B : [0, \infty) \rightarrow \mathbb{R}$, $B_0 = 0$, $B(t) = B_t$.

Facts about Brownian motion.

1) $t \mapsto B_t$ is continuous

2) $B_t \sim \mathcal{N}(0,t)$, $\sigma B_t \sim B_{\sigma^2 t} \sim \mathcal{N}(0,\sigma^2 t)$

3) $B_{t+s} - B_s \sim \mathcal{N}(0,t)$ (stationary increments)

4) B_t is independent of B_s for $0 \leq s < t$ (independent increments)

5) $-B_t$ is a Brownian motion

6) B_t is Hölder α continuous for all $0 < \alpha < 1/2$

SLE

- Stochastic Loewner Evolution (aka Schramm's LE) introduced by Oded Schramm in 1999
- developed by Lawler, Schramm, Werner and Rohde, Marshall

The idea: let U_t be a Brownian motion!

SLE with parameter κ is obtained by choosing $U_t = \sqrt{\kappa} B_t$ where B_t is a standard one dimensional Brownian motion.

Definition. SLE_κ in the upper half plane is the random collection of conformal maps g_t obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

It is not obvious that g_t^{-1} is well-defined at U_t so that the curve γ can be defined. The following theorem establishes this.

Theorem (Rohde-Schramm). *There exists a curve γ associated to SLE$_\kappa$ (at least for $\kappa \neq 8$).*

Think of $\gamma(t) = g_t^{-1}(\sqrt{\kappa} B_t)$.

SLE$_\kappa$ is the random collection of conformal maps g_t (complex analysts) or the curve $\gamma[0, t]$ being generated in $\mathbb{H}$ (probabilists)!

Although changing the variance parameter κ does not qualitatively change the behaviour of Brownian motion, it drastically alters the behaviour of SLE.

Properties of SLE

Fact.

- $0 < \kappa \leq 4$: $\gamma(t)$ can be defined and is a random, simple curve.
- $4 < \kappa < 8$: $\gamma(t)$ can be defined, but it is not a simple curve. It has double points, but does not cross itself!
- $\kappa > 8$: $\gamma(t)$ is a space filling curve! It has double points, but does not cross itself. Yet it is space-filling!!

Conjecture. *The Hausdorff dimension of the paths $\gamma(t)$ depends on κ.*

- $\dim_H(SLE_\kappa) = 1 + \frac{\kappa}{8}$ *for* $\kappa < 8$
- $\dim_H(SLE_\kappa) = 2$ *for* $\kappa > 8$

Of course $\dim_H(\mathrm{SLE}_\kappa) = 2$ for $\kappa > 8$ since it is space-filling.

References

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Steffen Rohde and Oded Schramm, *Basic Properties of SLE*, preprint, 2001.