Stirling's Formula:
An Application of Calculus

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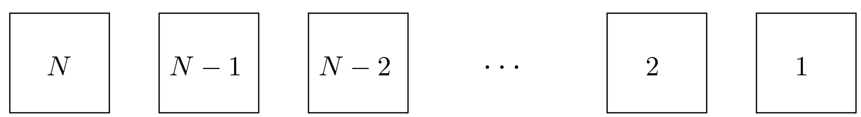
Permutations and Combinations

There are 2 *jumbles* of “AB” namely AB and BA.

There are 6 *jumbles* of “ABC” namely ABC, ACB, BAC, BCA, CAB, and CBA.

In general, there are $N!$ *jumbles* or *permutations* of N things. We can visualize this as putting objects into boxes:



$$N \times (N-1) \times (N-2) \times \cdots \times 2 \times 1 = N!$$

If there are N objects, then there are $\binom{N}{k}$ ways of choosing k of them:

$$\binom{N}{k} = \frac{N!}{k!\,(N-k)!}.$$

Growth of $N!$

We compute some values:

$1!$	1
$2!$	2
$3!$	6
$4!$	24
$5!$	120
$20!$	2.4329×10^{18}
$40!$	8.16×10^{47}
$100!$	9.33×10^{157}
$200!$	error

Obviously, $N!$ grows **very** quickly; much faster than exponential.

To compare, $e^{100} = 2.69 \times 10^{43}$.

An Approximation

Let $\mathfrak{S}(N) = \sqrt{2\pi}\, e^{-N} \; N^{N+\frac{1}{2}}$.

$\mathfrak{S}(5)$	118.0191680
$\mathfrak{S}(20)$	2.4228×10^{18}
$\mathfrak{S}(40)$	8.14×10^{47}
$\mathfrak{S}(100)$	9.32×10^{157}
$\mathfrak{S}(200)$	error

Stirling's Formula

Theorem:

$$\lim_{N\to\infty} \frac{N!}{\sqrt{2\pi}\, e^{-N}\ N^{N+\frac{1}{2}}} = 1$$

In other words, for large N,

$$N! \simeq \sqrt{2\pi}\, e^{-N}\ N^{N+\frac{1}{2}}.$$

Notation: $f(N) \simeq g(N)$ means that $f(N)/g(N) \to 1$ as $N \to \infty$.

Some History

Stirling's formula was discovered by Abraham de Moivre and published in "Miscellenea Analytica" in 1730. It was later refined, but published in the same year, by James Stirling in "Methodus Differentialis" along with other fabulous results. For instance, Stirling computes the area under the **Bell Curve**:

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

We will come back to this computation of Stirling ...

Aside: A Double Integral Computation

We compute $\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx$ via double integrals.

If $I = \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx$, then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx\right)^2 \\ &= \int_{-\infty}^{+\infty} e^{-y^2/2}\, dy \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(-x^2-y^2)/2}\, dx\, dy \end{aligned}$$

Changing to polar coordinates: $x = r\ \cos\theta$, $y = r\ \sin\theta$, $0 \le r < \infty$, $0 \le \theta < 2\pi$, and remembering the change-of-variables factor (Jacobian) $J = r$, we get

$$\begin{aligned} I^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{(-x^2-y^2)/2}\ dx\ dy \\ &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2/2}\ r\ dr\ d\theta \\ &= 2\pi \int_0^{+\infty} r\ e^{-r^2/2}\ dr \\ &= 2\pi \quad \text{make a substitution!} \end{aligned}$$

The Gamma Function Representation

For natural N, define the Gamma function $\Gamma(N)$ as

$$\Gamma(N) = \int_0^\infty e^{-t}\, t^{N-1}\, dt.$$

Proposition: $\Gamma(N+1) = N!$

Proof. To see this, let $u = t^N$, and $dv = e^{-t}\, dt$, and integrate by parts:

$$\begin{aligned}\int_0^\infty e^{-t}\, t^N\, dt &= -e^{-t}\, t^N|_0^\infty + N \int_0^\infty e^{-t}\, t^{N-1}\, dt \\ &= N \int_0^\infty e^{-t}\, t^{N-1}\, dt.\end{aligned}$$

Thus, we can iterate in N:

$$\begin{aligned} N\int_0^\infty e^{-t}\, t^{N-1}\, dt &= N(N-1)\int_0^\infty e^{-t}\, t^{N-2}\, dt \\ &= N(N-1)(N-2)\int_0^\infty e^{-t}\, t^{N-3}\, dt \\ &= N!\int_0^\infty e^{-t}\, t^{N-N}\, dt \\ &= N!\int_0^\infty e^{-t}\, dt \\ &= N! \end{aligned}$$

$\square$

Now, there is no reason to limit $\Gamma(N)$ to just NATURAL N. Indeed, the integral

$$\int_0^\infty e^{-t}\, t^{x-1}\, dt$$

converges for $0 < x < \infty$.

Note: When $x < 1$, both 0 and ∞ must be considered.

Definition: For $0 < x < \infty$, define

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

to be the Gamma function.

Thus, the Gamma function may be considered as the generalized factorial.

This is the natural way to consider "$x!$" for non-natural x.

Changing variables just as we did for $N!$ yields

Proposition: $\Gamma(x + 1) = x\, \Gamma(x)$.

Another change-of-variables reveals the identity:

Proposition: $\Gamma(1/2) = \int_{-\infty}^{\infty} e^{-x^2} \, dx$

Stirling showed that

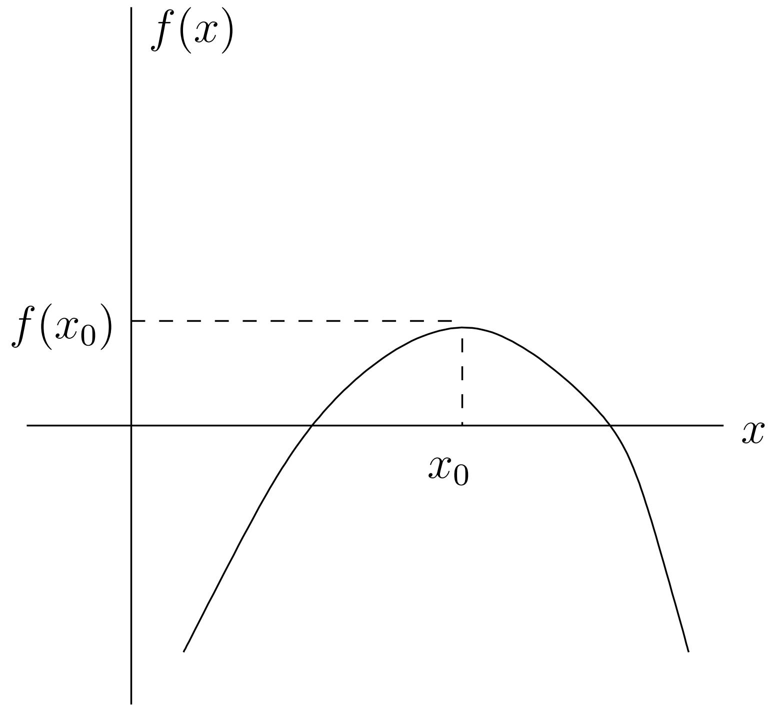
$$\Gamma(1/2) = \sqrt{\pi}.$$

This is equivalent to showing that the area under the bell curve is $\sqrt{2\pi}$.

CHECK! Change variables, and use the calculus fact about integrals of even functions over symmetric intervals.

Laplace's Method

Goal: Estimate $\int_{-\infty}^{\infty} e^{Nf(x)}\, dx$ for large N, where f looks like



It can be shown that for such a function, the main contribution to the integral comes from values of x near x_0.

In other words, write

$$\int_{-\infty}^{\infty} e^{Nf(x)}\, dx = \int_{(1-\epsilon)x_0}^{(1+\epsilon)x_0} e^{Nf(x)}\, dx + \int_{(1+\epsilon)x_0}^{\infty} e^{Nf(x)}\, dx + \int_{-\infty}^{(1-\epsilon)x_0} e^{Nf(x)}\, dx$$

where $\epsilon > 0$ is an arbitrary constant.

The contibutions from the second and third integral are asymptotically negligible.

That is, as $N \to \infty$,

$$\int_{(1+\epsilon)x_0}^{\infty} e^{Nf(x)}\, dx \to 0 \quad \text{and} \quad \int_{-\infty}^{(1-\epsilon)x_0} e^{Nf(x)}\, dx \to 0.$$

So, for large N, we have

$$\int_{-\infty}^{\infty} e^{Nf(x)}\, dx \simeq \int_{(1-\epsilon)x_0}^{(1+\epsilon)x_0} e^{Nf(x)}\, dx. \qquad (*)$$

Choose $\epsilon > 0$ small, and use a Taylor expansion around x_0 to obtain

$$f(x) \approx f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

since $f'(x_0) = 0$.

Note that $f''(x_0) < 0$, since f has a global maximum at x_0.

Substituting into $(*)$ yields

$$\int_{-\infty}^{\infty} e^{Nf(x)}\, dx \simeq e^{Nf(x_0)} \int_{(1-\epsilon)x_0}^{(1+\epsilon)x_0} e^{\frac{N}{2} f''(x_0)(x-x_0)^2}\, dx. \qquad (**)$$

Setting $u = \sqrt{-Nf''(x_0)}\,(x - x_0)$ and $du = \sqrt{-Nf''(x_0)}\, dx$ gives

$$(**) = \frac{e^{Nf(x_0)}}{\sqrt{-Nf''(x_0)}} \int_{-\epsilon\sqrt{-Nf''(x_0)}}^{\epsilon\sqrt{-Nf''(x_0)}} e^{-\frac{1}{2}u^2}\, du \simeq \frac{e^{Nf(x_0)}}{\sqrt{-Nf''(x_0)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du$$

$$= \frac{\sqrt{2\pi}\, e^{Nf(x_0)}}{\sqrt{-Nf''(x_0)}}.$$

Proof of Stirling's Formula

Recall that

$$N! = \int_0^\infty e^{-t}\ t^N\ dt = \int_0^\infty e^{-t+N\ln t}\ dt.$$

Changing variables with $t = Nx$ and $dt = N dx$ yields

$$N! = N^{N+1} \int_0^\infty e^{N(\ln x - x)}\ dx = N^{N+1} \int_0^\infty e^{Nf(x)}\ dx$$

where $f(x) = \ln x - x$.

Note: f is of the desired form with $x_0 = 1$, $f(x_0) = -1$, $f'(x_0) = 0$, $f''(x_0) = -1$.

Stirling's formula may now be derived easily from Laplace's method:

$$N! \simeq N^{N+1} \frac{\sqrt{2\pi}\ e^{Nf(x_0)}}{\sqrt{-Nf''(x_0)}} = N^{N+1} \frac{\sqrt{2\pi}\ e^{-N}}{N^{1/2}} = \sqrt{2\pi}\ e^{-N}\ N^{N+\frac{1}{2}}.$$