Green's function estimates with application to loop-erased random walk

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1 Green's functions on $\mathbb{C}$

Notation. We write any and all of x, y, z, w for points in $\mathbb{C}$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the interior of the unit disk. If x, $y \in \mathbb{D}$, let

$$g_{\mathbb{D}}(x, y) = \log \left| \frac{\overline{y}x - 1}{y - x} \right|$$

denote the Green's function for $\mathbb{D}$. Note $g_{\mathbb{D}}(0, x) = g_{\mathbb{D}}(x) = -\log|x|$, and $g_{\mathbb{D}}(x, y) = g_{\mathbb{D}}(y, x)$.

Suppose $D \subsetneq \mathbb{C}$ is simply connected and $0 \in D$. By the Riemann mapping theorem, $\exists!$ conformal transformation $\psi_D(z) : D \to \mathbb{D}$ with $\psi_D(0) = 0$, $\psi_D'(0) > 0$. In this case, the Green's function for D is

$$g_D(z, w) = g_{\mathbb{D}}(\psi_D(z), \psi_D(w)) \text{ for } z, w \in D. \qquad (*)$$

Conversely, if we write $g_D(z) = g_D(z, 0) = g_D(0, z)$, then

$$\psi_D(z) = \exp\{-g_D(z) + i\theta_D(z)\}.$$

In other words, determining the Green's function for a simply connected, proper subset of $\mathbb{C}$ is equivalent to finding the Riemann mapping function of that domain onto the unit disk.

Equivalently, we can formulate the Green's function for D in terms of BM. Suppose B_t is a standard BM in $\mathbb{C}$ and $T_D = \inf\{t : B_t \notin D\}$.

If $x \in D$, we can define $g_D(x, \cdot)$ as the unique harmonic function on $D \setminus \{x\}$, vanishing on ∂D, with $g_D(x, y) = -\log|x - y| + O(1)$ as $|x - y| \to 0$. From this description we have for $x \neq y \in D$, $g_D(x, y) = \mathbf{E}^x[\log|B(T_D) - y|] - \log|x - y|$. In particular, if $0 \in D$, then

$$g_D(x) = \mathbf{E}^x[\log|B(T_D)|] - \log|x| \text{ for } x \in D.$$

2 Green's functions on $\mathbb{Z}^2$

Denote by $\mathcal{A}$, the set of all finite, simply connected $A \subseteq \mathbb{Z}^2$ containing the origin. Let $\text{inrad}(A) = \text{dist}(0, \partial A) = \inf\{|z| : z \in \mathbb{Z}^2 \setminus A\}$.

By $\mathcal{A}^n$, we mean those $A \in \mathcal{A}$ with $\text{inrad}(A) \in [n, 2n]$.

There are three reasonable ways to define the "boundary" of A.

- *(outer) boundary:* $\partial A := \{y \in \mathbb{Z}^2 \setminus A : |y - x| = 1 \text{ for some } x \in A\}$,
- *inner boundary:* $\partial_i A := \partial(\mathbb{Z}^2 \setminus A) = \{x \in A : |y - x| = 1 \text{ for some } y \in \mathbb{Z}^2 \setminus A\}$,
- *edge boundary:* $\partial_e A := \{(x, y) : x \in A,\, y \in \mathbb{Z}^2 \setminus A,\, |x - y| = 1\}$.

Suppose S_j is a SRW on $\mathbb{Z}^2$, $S_0 = 0$. If $\tau_A = \min\{j \geq 0 : S_j \notin A\}$, then

$$G_A(x, y) = \mathbf{E}^x[\sum_{j=0}^{\tau_A - 1} 1\{S_j = y\}] = \sum_{j=0}^{\infty} \mathbf{P}^x\{S_j = y, \tau_A > j\}$$

denotes the Green's function for A (for SRW), i.e., the expected number of visits from x to y before exiting A.

Let $G_A(x) = G_A(x, 0) = G_A(0, x)$. It is known that

$$G_A(x) = \mathbf{E}^x[a(S(\tau_A))] - a(x) \text{ for } x \in A.$$

where a is the potential kernel for SRW defined by

$$a(x) = \lim_{m \to \infty} \sum_{j=0}^{m} [\mathbf{P}^0\{S_j = 0\} - \mathbf{P}^x\{S_j = 0\}].$$

It is also known that as $|x| \to \infty$,

$$a(x) = \frac{2}{\pi} \log |x| + k_0 + o(|x|^{-3/2}) \tag{1}$$

where $k_0 = (2\gamma + 3 \ln 2)/\pi$ and γ is Euler's constant. Stronger results are known. The asymptotic expansion of $a(x)$ given in Fukai-Uchiyama shows that the error is $O(|x|^{-2})$.

3 Riemann Mapping $\tilde{A}$ to $\mathbb{D}$

To each $A \in \mathcal{A}^n$ we associate a domain $\tilde{A} \subseteq \mathbb{C}$ in the following way:

$$\tilde{A} \cup \partial \tilde{A} = \bigcup_{x \in A} \mathcal{S}_x,$$

where $\mathcal{S}_x$ is the closed square of side one centered at x whose sides are parallel to the coordinate axes. Also, note that $\tilde{A} \subseteq \mathbb{C}$ is simply connected iff $A \subseteq \mathbb{Z}^2$ is simply connected.

That is, put a unit square about each point in A. The interior of the union of these squares is $\tilde{A}$.

Let $\psi_A(z)$ be the conformal transformation of $\tilde{A}$ onto $\mathbb{D}$ with $\psi_A(0) = 0$, $\psi_A'(0) > 0$.

If we let $g_A(x, y) := g_{\tilde{A}}(x, y)$ be the Green's function for $\tilde{A}$ (for BM), and set $g_A(x) = g_A(0, x)$, then we can write the Riemann map as

$$\psi_A(x) = \exp\{-g_A(x) + i\theta_A(x)\}.$$

By ($*$), we then have that the Green's function for $\tilde{A}$ is given by

$$g_A(x, y) = g_{\mathbb{D}}(\psi_A(x), \psi_A(y)) = \log \left| \frac{\overline{\psi_A(y)}\psi_A(x) - 1}{\psi_A(y) - \psi_A(x)} \right|. \tag{2}$$

4 The Koebe one-quarter theorem and its consequences

The following are corollaries to the Koebe one-quarter theorem, and the growth and distortion theorems. There are proofs to similar results in Lawler's SLE notes.

Corollary 1. *If $A \in \mathcal{A}^n$, then* $-\log \psi_A'(0) = \log n + O(1)$.

Corollary 2. *If $A \in \mathcal{A}^n$ and $|x| \leq n/16$, then*

$$\psi_A(x) = x\psi_A'(0) + |x|^2 O(n^{-2}),$$

and

$$g_A(x) + \log |x| = -\log \psi_A'(0) + |x| O(n^{-1}).$$

Note that for $x = 0$ the left hand side is defined by the limit and the error term on the right hand side disappears.

5 Beurling estimates

Suppose B_t is a BM in $\mathbb{C}$, and $T_A := T_{\tilde{A}} = \inf\{t : B_t \notin \tilde{A}\}$. From the Beurling projection theorem, we have

Corollary (Beurling Estimate)**.** *There is a constant $c < \infty$ such that if $x \in \tilde{A}$, then for all $r > 0$,*

$$\mathbf{P}^x\{|B(T_A) - x| > r \operatorname{dist}(x, \partial\tilde{A})\} \le c\, r^{-1/2}.$$

From this, we can deduce

$$g_A(x) \le c\, n^{-1/2} \operatorname{dist}(x, \partial\tilde{A})^{1/2}, \quad A \in \mathcal{A}^n, \quad |x| \ge n/4$$

so that

$$g_A(x) \le c n^{-1/2} \text{ for } x \in \partial_i A.$$

Hence, $\psi_A(x) = \exp\{i\theta_A(x)\} + O(n^{-1/2})$, and if $x, y \in \partial_i A$,

$$|\psi_A(x) - \psi_A(y)| = [1 - \cos(\theta_A(x) - \theta_A(y))]^{-1} + O(n^{-1/2}).$$

If $z \in \partial A$, we define $\theta_A(z)$ to be the average of $\theta_A(x)$ over all $x \in A$ with $|x - z| = 1$. The Beurling estimate and a simple Harnack principle show that

$$\theta_A(z) = \theta_A(x) + O(n^{-1/2}), \quad (x, z) \in \partial_e A.$$

There are similar results in the discrete case. Suppose S_n is SRW on $\mathbb{Z}^2$, and $\tau_A = \min\{j \ge 0 : S_j \notin A\}$. From the discrete Beurling projection theorem, we have

Corollary (Discrete Beurling Estimate)**.** *There is a constant c such that if $x \in A$, then for all $r > 0$,*

$$\mathbf{P}^x\{|S(\tau_A) - x| > r \operatorname{dist}(x, \partial A)\} \le c\, r^{-1/2}.$$

Thus, we can deduce

$$G_A(x) \le c\, n^{-1/2} \operatorname{dist}(x, \partial A)^{1/2}, \quad A \in \mathcal{A}^n, \quad |x| \ge n/4,$$

so that

$$G_A(x) \le c n^{-1/2} \text{ for } x \in \partial_i A. \qquad (**)$$

If $A \in \mathcal{A}$ and $0 \ne x \in \partial_i A$, then

$$G_A(0) = G_{A\setminus\{x\}}(0) + \frac{G_A(x)^2}{G_A(x,x)}.$$

If we replace $A \setminus \{x\}$ with the connected component of $A \setminus \{x\}$ containing the origin, then by $(**)$

$$G_A(0) - G_{A\setminus\{x\}}(0) \le G_A(x)^2 \le c\, n^{-1}, \quad A \in \mathcal{A}^n,\ x \in \partial_i A.$$

6 Main Result

Theorem 1. *There exists a decreasing sequence $\varepsilon_n \downarrow 0$ such that if $A \in \mathcal{A}^n$,*

$$G_A(0) = -\frac{2}{\pi} \log \psi_A'(0) + k_0 + O(\varepsilon_n^3),$$

where k_0 is the constant in (1). Moreover, if $x, y \in \partial_i A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$,

$$G_A(x,y) = \frac{(\pi/2)\, G_A(x)\, G_A(y)}{1 - \cos(\theta_A(x) - \theta_A(y))}\, [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})]. \tag{3}$$

There are two parts to this theorem, and we handle them as separate propositions. First, we estimate $G_A(x,y)$ for x, y not too close to the boundary, which means that $\psi_A(x)$ and $\psi_A(y)$ are not close to $\partial\mathbb{D}$. Our second proposition will give estimates for x and y close to the boundary provided that they are not too close to each other.

Note that $\varepsilon_n = n^{-1/48} \log^{2/3} n$.

7 Green's function estimates

7.1 Estimates away from the boundary

Goal. Estimate $G_A(x,y)$ for x, y not too close to the boundary.

This follows from a result that says the place SRW leaves A is close to the place BM leaves $\tilde{A}$. The proof uses two facts: a strong approximation result, which will tell us that when BM hits the boundary the SRW is close to the BM, and the Beurling estimates, which tells us that if a BM or a SRW is close to the boundary then it will hit it soon.

Proposition 1. *There exists a constant c such that for every n, B_t and S_t can be defined on the same probability space so that if $A \in \mathcal{A}^n$, $1 < r \leq n^{20}$, and $x \in A$ with $|x| \leq n^3$,*

$$\mathbf{P}^x\{|B(T_A) - S(\tau_A)| \geq c\, r\, \log n\} \leq c\, r^{-1/2}.$$

Proof. We will use the following fact that can be easily derived from the strong approximation theorem [1]: there exists a constant c_1 such that a SRW S_t and a BM B_t can be defined at the origin on the same probability space so that, except for an event of probability $O(n^{-10})$,

$$|B_t - S_t| \leq c_1\, \log n, \qquad 0 \leq t \leq \sigma_n,$$

where σ_n^1 (resp., σ_n^2) is the first time the BM (resp., SRW) gets distance n^8 from its starting point and $\sigma_n = \max\{\sigma_n^1, \sigma_n^2\}$. For any given n, let (B_t, S_t) be defined as above. Let

$$T_A' = \inf\{t \geq 0 : \mathrm{dist}(B_t, \partial\tilde{A}) \leq 2c_1 \log n\},\ \tau_A' = \inf\{t \geq 0 : \mathrm{dist}(S_t, \partial A) \leq 2c_1 \log n\},$$

and let V_1, V_2, V_3 be the events

$$V_1 = \{ \sup_{0 \le t \le \sigma_n} |B_t - S_t| > c_1 \ \log n \},$$

$$V_2 = \{ \sup_{T_A' \le t \le T_A} |B_t - B_{T_A'}| \ge r \ \log n \}, \quad V_3 = \{ \sup_{\tau_A' \le t \le \tau_A} |S_t - S_{\tau_A'}| \ge r \ \log n \}.$$

By the Beurling projection theorems and the strong Markov property, we can see that $\mathbf{P}(V_2 \cup V_3) \le c r^{-1/2}$. Also, $\mathbf{P}(V_1) \le n^{-10} \le r^{-1/2}$. But on the complement of $V_1 \cup V_2 \cup V_3$, $|B(T_A) - S(\tau_A)| \le (r + c_1) \log n$. □

Corollary 3. *There exists a c such that if $A \in \mathcal{A}^n$ and $|x| \le n^2$,*

$$| \ \mathbf{E}^x[\log |B(T_A)|] - \mathbf{E}^x[\log |S(\tau_A)|] \ | \le c \, n^{-1/3} \ \log n.$$

For any $A \in \mathcal{A}^n$, let $A^{*,n}$ be the set

$$A^{*,n} = \{ x \in A : g_A(x) \ge n^{-1/16} \}.$$

The choice of $1/16$ for the exponent is somewhat arbitrary, and slightly better estimates might be obtained by choosing a different exponent. However, since we do not expect the error estimate derived here to be optimal, we will just make this definition.

Corollary 4. *If $A \in \mathcal{A}^n$, and $x \in A^{*,n}, y \in A$, then*

$$G_A(x,y) = (2/\pi) \, g_A(x,y) + k_{y-x} + O(n^{-7/24} \log n)$$

where $k_x = k_0 + (2/\pi) \ \log |x| - a(x)$. Note that $|k_x| \le c|x|^{-3/2}$.

7.2 Estimates near the boundary

Goal. Estimate $G_A(x,y)$ for x, y close to the boundary, but not close to each other.

Let $J_{x,n} = \{ z \in A : |\psi_A(z) - \exp\{i\theta_A(x)\}| \ge n^{-1/16} \log^2 n \}$.

Proposition 2. *Suppose $A \in \mathcal{A}^n$ and $x \in A \setminus A^{*,n}$, $y \in J_{x,n}$. Then,*

$$G_A(x,y) = G_A(x) \ \frac{1 - |\psi_A(y)|^2}{|\psi_A(y) - e^{i\theta_A(x)}|^2} \ [\ 1 + O(\frac{n^{-1/16} \log n}{|\psi_A(y) - e^{i\theta_A(x)}|}) \ ], \quad y \in A^{*,n},$$

$$G_A(x,y) = \frac{(\pi/2) \ G_A(x) \ G_A(y)}{1 - \cos(\theta_A(x) - \theta_A(y))} \ [\ 1 + O(\frac{n^{-1/16} \log n}{|\theta_A(y) - \theta_A(x)|}) \ ], \quad y \in A \setminus A^{*,n}.$$

There is nothing surprising about the leading term. The following estimates can be derived easily from (2). If $z = \psi_A(x) = (1-r)e^{i\theta}$, $z' = \psi_A(y) \in \mathbb{D}$ with $|z - z'| \geq r$, then

$$g_A(x,y) = g_{\mathbb{D}}(z,z') = \frac{g_{\mathbb{D}}(z)\ (1-|z'|^2)}{|z' - e^{i\theta}|^2}\ [1 + O(\frac{r}{|z-z'|})].$$

Similarly, if $z' = \psi_A(y) = (1-r')e^{i\theta'}$ with $r \geq r'$ and $|z - z'| \geq r$,

$$g_A(x,y) = g_{\mathbb{D}}(z,z') = \frac{g_{\mathbb{D}}(z)\ g_{\mathbb{D}}(z')}{1-\cos(\theta-\theta')}\ [1 + O(\frac{r}{|\theta-\theta'|})].$$

The proposition essentially says that these relations are valid (at least in the dominant term) if we replace g_A with $(\pi/2)\, G_A$.

The hardest part of the proof is a lemma that states that if the SRW starts at a point x with $\psi_A(x)$ near $\partial\mathbb{D}$, then, given that the walk does not leave A, $\psi_A(S_j)$ moves a little towards the center of the disk before its argument changes too much.

7.3 An estimate for hitting the boundary

In order to prove Proposition 2, we will need a lemma which states roughly that if BM has a good chance of exiting $\partial\tilde{A}$ at a particular collection of segments $\tilde{V}$ there is also a good chance that SRW exits A at the corresponding set V in ∂A. Specifically, let A be any finite, connected subset of $\mathbb{Z}^2$, not necessarily simply connected, and let $V \subseteq \partial A$. For every $y \in V$, consider the collection of edges containing y, namely $\mathcal{E}_y := \{(x,y) \in \partial_e A\}$; let $\mathcal{E}_V = \cup_{y\in V}\mathcal{E}_y$ where $\ell_{x,y}$ is the perpendicular line segment of length 1intersecting (x,y) in the midpoint, and define

$$\tilde{V} = \bigcup_{(x,y)\in\mathcal{E}_V} \ell_{x,y}.$$

Let $T = T_A$, $\tau = \tau_A$ be as before, and let

$$f(x) = f_{A,V}(x) = \mathbf{P}^x\{S_{\tau_A} \in V\}, \quad \tilde{f}(x) = \tilde{f}_{\tilde{A},\tilde{V}}(x) = \mathbf{P}^x\{B_{T_A} \in \tilde{V}\}.$$

Let Δ denote the usual Laplacian in $\mathbb{C}$ and call a function h harmonic at x if $\Delta h(x) = 0$. Let L denote the discrete Laplacian

$$Lh(x) = \frac{1}{4}\sum_{|x-y|=1}(h(y) - h(x))$$

and call h discrete harmonic at x if $Lh(x) = 0$. Note that $\tilde{f}$ is harmonic in $\tilde{A}$ and f is discrete harmonic in A. It follows from Taylor series and uniform bounds on derivatives of harmonic functions that if $r > 1$ and h is harmonic on $\{z \in \mathbb{C} : |z| < r\}$, then

$$|Lh(0)| \leq \|h\|_\infty\, O(r^{-3}).$$

Lemma 1. *For every $\varepsilon > 0$ there is a $\delta > 0$, such that if A is a finite connected subset of $\mathbb{Z}^2$, $V \subseteq \partial A$, and $x \in A$ with $\tilde{f}(x) \geq \varepsilon$, then $f(x) \geq \delta$.*

We first note for every $n < \infty$, there is a $\delta' = \delta'(n) > 0$ such that the lemma holds for all A of cardinality at most n and all $\varepsilon > 0$. This is because f, $\tilde{f}$ are strictly positive (assuming V is nonempty) and the collection of connected subsets of $\mathbb{Z}^2$ containing the origin of cardinality at most n is finite. Hence we can choose

$$\delta'(n) = \min \mathbf{P}^x\{S_{\tau_A} = (y, z)\},$$

where the minimum is over all finite connected A of cardinality at most n, over all $x \in A$, and over all $(y, z) \in \partial_e A$.

8 Application to loop-erased random walk

In this section we combine the Green's function estimate of Theorem 1 with an identity of S. Fomin, to give an estimate for the probability of a particular event dealing with loop-erased random walk.

8.1 Hitting probabilities

If $x \in A$, $y \in \partial A$, let $H_A(x, y) = \mathbf{P}^x\{S(\tau_A) = y\}$ be the probability that SRW starting at x exits A at y. Then a simple last exit decomposition gives:

Fact 1. $H_A(x, y) = \frac{1}{4} \sum_{(z,y)\in\partial_e A} G_A(x, z)$

Proof.

$$\begin{aligned} \mathbf{P}^x\{S(\tau_A) = y\} &= \sum_{(z,y)\in\partial_e A} \sum_{k=1}^{\infty} \mathbf{P}^x\{S_k = y | S_{k-1} = z, \tau_A = k\}\, \mathbf{P}^x\{S_{k-1} = z, \tau_A = k\} \\ &= \sum_{(z,y)\in\partial_e A} \sum_{k=1}^{\infty} \frac{1}{4}\, \mathbf{P}^x\{S_{k-1} = z, \tau_A = k\} = \frac{1}{4} \sum_{(z,y)\in\partial_e A} G_A(x, z) \end{aligned}$$

□

Using this we can derive the following from (3):

Corollary 5. *If $A \in \mathcal{A}^n$, $x \in \partial_i A$, $y \in \partial A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then*

$$H_A(x, y) = \frac{(\pi/2)\, G_A(x)\, H_A(0, y)}{1 - \cos(\theta_A(x) - \theta_A(y))}\, [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})]. \tag{4}$$

Similarly, if $x \in \partial A$, $y \in \partial A$, let $H_A(x, y) = \mathbf{P}^x\{S(\tau_A) = y | S_1 \in A\}$ be the probability that a SRW starting at x takes its first step into A and then exits A at y.

Fact 2. $H_A(x,y) = \frac{1}{4} \sum_{(z,x)\in\partial_e A} H_A(z,y)$

Proof.

$$H_A(x,y) = \sum_{(z,x)\in\partial_e A} \mathbf{P}^x\{S(\tau_A) = y | S_1 = z\}\, \mathbf{P}^x\{S_1 = z\} = \frac{1}{4} \sum_{(z,x)\in\partial_e A} \mathbf{P}^z\{S(\tau_A) = y\} \quad \square$$

Combining this with (3) and (4) we get:

Corollary 6. *If $A \in \mathcal{A}^n$, $x, y \in \partial A$ with $|\theta_A(x) - \theta_A(y)| \geq \varepsilon_n$, then*

$$H_A(x,y) = \frac{(\pi/2)\, H_A(0,x)\, H_A(0,y)}{1 - \cos(\theta_A(x) - \theta_A(y))}\, [1 + O(\frac{\varepsilon_n^3}{|\theta_A(x) - \theta_A(y)|})]. \tag{5}$$

8.2 Loop-erased random walk and Fomin's identity

If $S = [S(0), S(1), \ldots, S(m)]$ is a SRW path of length m, let $\Lambda(S)$ be the loop-erased part of S, which can be constructed as follows. If S is already self-avoiding, set $\Lambda(S) = S$. Otherwise, let $s_0 = \max\{j : S(j) = S(0)\}$, and for $i > 0$, let $s_i = \max\{j : S(j) = S(s_{j-1} + 1)\}$. If we let $n = \min\{i : s_i = m\}$, then $\Lambda(S) = [S(s_0), S(s_1), \ldots, S(s_n)]$. Suppose that $A \in \mathcal{A}^n$ and $x^1, x^2, \ldots, x^N \in \partial A$. Let $S^1, S^2, \ldots, S^N$ be independent SRW starting at $x^1, x^2, \ldots, x^N$, respectively, and let

$$\tau_A^k := \min\{j > 0 : S_j^k \notin A\}.$$

Let $L^k = \Lambda(S^k)$ be the loop erasure of the path $[S^k(0) = x^k, S^k(1), \ldots, S^k(\tau_A^k)]$, and let $\mathcal{E} = \mathcal{E}(x^1, \ldots, x^N, y^1, \ldots, y^N; A)$ be the event that

- $S^k(\tau_A^k) = y^k, \quad k = 1, \ldots, N$, and
- $S^k[0, \tau_A^k] \cap \{L^1 \cup \cdots \cup L^{k-1}\} = \emptyset, \quad k = 2, \ldots, N.$

Theorem (Fomin [2, Theorem 7.5])**.** *If $\mathbf{H}_A = [H_A(x^k, y^\ell)]$ is the $N \times N$ hitting matrix*

$$\mathbf{H}_A = \begin{bmatrix} H_A(x^1, y^1) & \cdots & H_A(x^1, y^N) \\ \vdots & \ddots & \vdots \\ H_A(x^N, y^1) & \cdots & H_A(x^N, y^N) \end{bmatrix}$$

then

$$\mathbf{P}\{\mathcal{E}\} = \det[\mathbf{H}_A].$$

Combining Fomin's Theorem with (5) yields the following:

Theorem 2. *Suppose that $A \in \mathcal{A}^n$ and $x^1, \ldots, x^N, y^1, \ldots, y^N \in \partial A$ with*

$$\delta = \min_{1 \leq k, \ell \leq N} \{|\theta_A(x^k) - \theta_A(y^\ell)|\} \geq \varepsilon_n.$$

Let $\varphi_A(x^k, y^\ell) = [1 - \cos(\theta_A(x^k) - \theta_A(y^\ell))]^{-1}$. If $\mathcal{E}$ is the event defined as above, then

$$\mathbf{P}\{\mathcal{E}\} = (\pi/2)^N \, [\prod_{k=1}^{N} H_A(0, x^k)] \, [\prod_{\ell=1}^{N} H_A(0, y^\ell)] \; \det[\mathbf{\Phi}_A] \, [1 + O(\varepsilon_n^3 \delta^{-1})]$$

where $\mathbf{\Phi}_A$ is the $N \times N$ matrix $\mathbf{\Phi}_A = [\varphi_A(x^k, y^\ell)]$.

References

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