A rate of convergence for loop-erased random walk to SLE(2)

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Conformal maps from probability to physics
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Introduction

The Schramm-Loewner evolution with parameter κ (SLE$_\kappa$) was introduced in 1999 by Oded Schramm while considering possible scaling limits of loop-erased random walk.

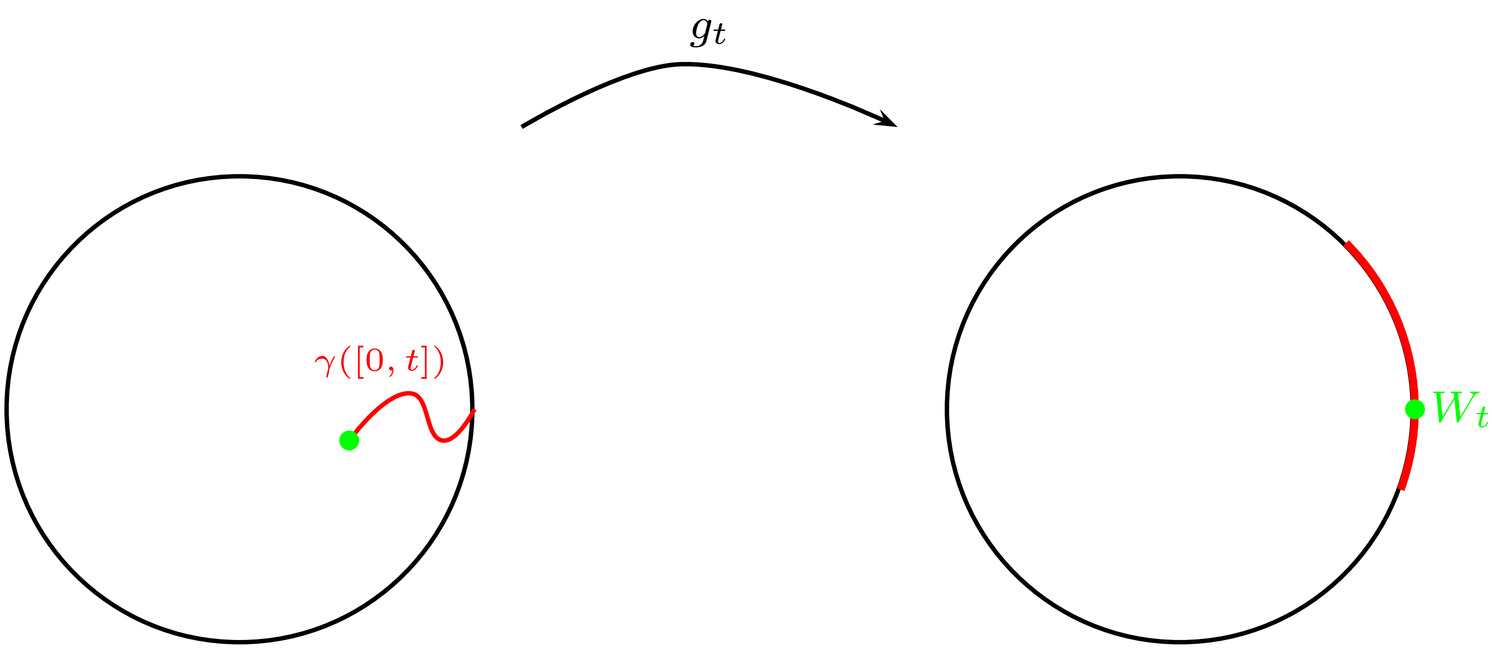
Since then, it has successfully been used to study various other lattice models from two-dimensional statistical mechanics including percolation, uniform spanning trees, self-avoiding walk, and the Ising model.

Crudely, one defines a discrete interface on the $1/N$-scale lattice and then lets $N \to \infty$. The limiting continuous "interface" is an SLE.

In "Conformal invariance of planar loop-erased random walks and uniform spanning trees" (AOP 2004), Lawler, Schramm, and Werner showed that the scaling limit of loop-erased random walk is SLE with parameter $\kappa = 2$. The proof is qualitative and no rate of convergence immediately follows from it.

Schramm's ICM 2006 Problem 3.1: "Obtain reasonable estimates for the speed of convergence of the discrete processes which are known to converge to SLE."

Review of Radial SLE



Reparametrize γ so that

$$g_t'(0) = e^t.$$

This is the capacity parametrization.

Review of Radial SLE (cont)

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{D}_t \to \mathbb{D}$, can be described by the Loewner equation.

For $z \in \mathbb{D}$ with $z \notin \gamma[0, \infty]$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z,$$

where

$$W_t = \lim_{z \to \gamma(t)} g_t(z).$$

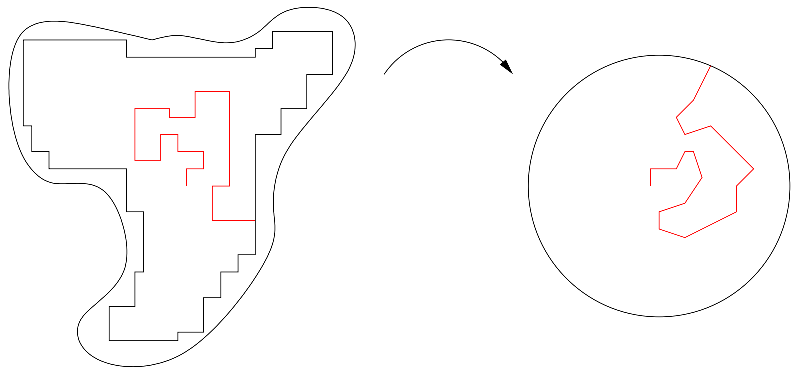
We call W the driving function of the curve γ.

The **radial Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard parametrization** (or simply SLE$_\kappa$) is the random collection of conformal maps $\{g_t,\, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{e^{i\sqrt{\kappa}B_t} + g_t(z)}{e^{i\sqrt{\kappa}B_t} - g_t(z)}, \quad g_0(z) = z. \tag{LE}$$

where B_t is a standard one-dimensional Brownian motion.

From LERW to SLE



- Let $D \ni 0$ be a simply connected planar domain with $\frac{1}{n}\mathbb{Z}^2$ grid domain approximation $D_n \subset \mathbb{C}$. A grid domain is a domain whose boundary is a union of edges of the scaled lattice. That is, D_n is the connected component containing 0 in the complement of the closed faces of $n^{-1}\mathbb{Z}^2$ intersecting ∂D. Note that D_n is simply connected.
- $\psi_{D_n} : D_n \to \mathbb{D}$, $\psi_{D_n}(0) = 0, \psi'_{D_n}(0) > 0$.
- γ_n: time-reversed LERW from 0 to ∂D_n (on $\frac{1}{n}\mathbb{Z}^2$).
- $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ is a path in $\mathbb{D}$. Parameterize by capacity.
- $W_n(t) = W_0 e^{i\vartheta_n(t)}$: the Loewner driving function for $\tilde{\gamma}_n$.

L-S-W Prove Convergence of the Driving Processes

Theorem (Lawler-Schramm-Werner, 2004). Let $\mathcal{D}$ be the set of simply connected grid domains with $0 \in D, D \neq \mathbb{C}$. For every $T > 0, \varepsilon > 0$, there exists $n = n(T, \varepsilon)$ such that if $D \in \mathcal{D}$ with inrad$(D) > n$, then there exists a coupling between loop-erased random walk γ from ∂D to 0 in D and Brownian motion B started uniformly on $[0, 2\pi]$ such that

$$\mathbf{P}\left\{ \sup_{0 \le t \le T} |\theta(t) - B(2t)| > \varepsilon \right\} < \varepsilon,$$

where $\theta(t)$ satisfies $W(t) = W(0)e^{i\theta(t)}$ and $W(t)$ is the driving process of γ in Loewner's equation.

This is "a kind of of convergence" of LERW to SLE_2, and leads (without too much difficulty) to the stronger convergence (which we won't discuss) of paths with respect to the Hausdorff metric.

L-S-W then use this result to establish convergence of paths with respect to the metric that identifies curves modulo reparametrization (c.f., Aizenman-Burchard).

Statement of Main Result

Our main result provides a rate for the convergence of the driving processes.

Theorem (Beneš-Johansson-K, 2009). Let $0 < \epsilon < 1/36$ be fixed, and let D be a simply connected domain with inrad$(D) = 1$. For every $T > 0$ there exists an $n_0 < \infty$ depending only on T such that whenever $n > n_0$ there is a coupling of γ_n with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that

$$\mathbf{P}\left(\sup_{0 \leq t \leq T} |W_n(t) - e^{iB(2t)}| > n^{-(1/36-\epsilon)} \right) < n^{-(1/36-\epsilon)}.$$

Recall that

$$W_n(t) = W_n(0)e^{i\vartheta_n(t)}, \quad t \geq 0,$$

denotes the Loewner driving function for the curve $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ parameterized by capacity.

Ideas of Proof

L-S-W follow the "three main steps" to proving convergence of the driving processes.

1. Find a discrete martingale observable for the LERW path. Prove that it converges to something conformally invariant.
2. Use Step 1 together with the Loewner equation to show that the Loewner driving function for the LERW is almost a martingale with "correct" (conditional) variance.
3. Use Step 2 and Skorokhod embedding to couple the Loewner driving function for the LERW with a Brownian motion and show that they are uniformly close with high probability.

To obtain a rate we have to re-examine the steps to find explicit bounds on error terms.

Some Notation

$D \neq \mathbb{C}$ is a simply connected grid domain containing the origin

$V = V(D)$ is the set of vertices contained in D

$\psi_D : D \to \mathbb{D}$ with $\psi_D(0) = 0$, $\psi'_D(0) > 0$

If D is a simply connected domain with a Jordan boundary, it is well-known that ψ_D can be extended continuously to the boundary so that if $u \in \partial D$, then $\psi_D(u) = e^{i\theta_D(u)} \in \partial\mathbb{D}$.

Our grid domains, however, will be "Jordan minus a slit." This means that a boundary point of D may correspond under conformal mapping to several points on the boundary of the unit disk. To avoid using prime ends, we adopt the L-S-W convention of viewing the boundary of $\mathbb{Z}^2 \cap D$ as pairs (u, e) of a point $u \in \partial D \cap \mathbb{Z}^2$ and an incident edge e.

We write $V_\partial(D)$ for the set of such pairs, and if $v \in V_\partial(D)$, then the notation $\psi_D(v)$ means $\lim_{z\to u} \psi_D(z)$ along e, and this limit always exists.

Step 1. A Martingale Observable

The martingale observable used by Lawler-Schramm-Werner is the discrete Poisson kernel.

Fix n and $z \in V(D_n)$ and let

$$M_k = M_k(z) := \frac{H_k(z, \gamma_n(k))}{H_k(0, \gamma_n(k))}, \quad k \geq 0.$$

If $x \in D_n \setminus \gamma_n[0,k]$, then $H_k(x, \gamma_n(k))$ denotes the probability that simple random walk started at x exits the slit domain $D_n \setminus \gamma_n[0,k]$ at $\gamma_n(k)$, i.e., discrete harmonic measure.

One can show that M_k is a martingale with respect to $\gamma_n[0,k]$, and for fixed k, $M_k(z)$ is discrete harmonic.

Discrete and Continuous Poisson Kernel

The next step is to show that for appropriate z when n is large, the discrete and continuous Poisson kernel are close:

$$\frac{H_k(z,\gamma_n(k))}{H_k(0,\gamma_n(k))} \approx \frac{1-|\psi_k(z)|^2}{|\psi_k(z)-\psi_k(\gamma_n(k))|^2}$$

with explicit error terms (in terms of the lattice scale $1/n$). Here $\psi_k : D_n \setminus \gamma_n[0,k] \to \mathbb{D}$.

K-Lawler, 2005, did a similar estimate, but worked in a slightly different setting (union of squares domains) with simply connected Jordan domains only. The goal then was different and that result was not optimal for that setting.

Those ideas, however, can be modified to apply to the present setting, and optimality can be attempted. (K-L studied two arbitrary points not too close to each other; B-J-K need one "interior" point and one point "next to the boundary.")

Discrete and Continuous Poisson Kernel

Theorem (Beneš-Johansson-K, 2009). Let $0 < \epsilon < 1/6$ and let $0 < \rho < 1$ be fixed. Suppose that D is a grid domain satisfying $n \le \mathrm{inrad}(D) \le 2n$. Furthermore, suppose that $x \in D \cap \mathbb{Z}^2$ with $|\psi_D(x)| \le \rho$ and $u \in V_\partial(D)$. If both x and u are accessible by a simple random walk starting from 0, then

$$\frac{H_D(x,u)}{H_D(0,u)} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - \psi_D(u)|^2} \cdot [\, 1 + O(n^{-(1/6-\epsilon)})\,].$$

Note. The condition $|\psi_D(x)| \le \rho$ ensures that x is an "interior" point.

Proving the Closeness of the Poisson Kernels

The proof relies on the fact that for all $\varepsilon > 0$, we can find $\delta > 0$ such that if

- D is a(n appropriate) grid domain,
- $E \subset \partial D$ is a union of edges of $\mathbb{Z}^2$,
- x is far enough from ∂D,

then

$$H(x) \geq \varepsilon \Rightarrow h(x) \geq \delta, \qquad (*)$$

where $h(x) = h_D(x, E)$ is discrete harmonic measure of E from x and $H(x)$ is its continuous analogue.

The implication in $(*)$ is generally not satisfied by grid domains (problems arise with fjords or channels).

Proving the Closeness of the Poisson Kernels

To get around this problem, one can cut off anything in the domain that isn't accessible by random walk started at x by creating a Union of Big Squares (UBS) domain D_0, which is the union of squares of side length 2 centred at the points of $V_0(D)$, i.e., those vertices in the connected component containing 0.

Beurling's theorem implies that if the Poisson kernels are close in D_0, they are close in D.

Specifically, if $n \leq \text{inrad}(D) \leq 2n$, then

$$\partial\psi_D(D_0) \subset \mathcal{A}(1 - cn^{-1/2}, 1).$$

The conformal map from $\psi(D_0)$ to $\mathbb{D}$ is almost the identity and one can show that

$$\psi_{D_0}(x) = \psi_D(x) + O(n^{-1/2}\log n)$$

and

$$e^{i\theta_{D_0}(u)} = e^{i\theta_D(u)} + O(n^{-1/4}).$$

Proving the Closeness of the Poisson Kernels

$$\psi_{D_0}(x) = \psi_D(x) + O(n^{-1/2}\log n) \quad \text{and} \quad e^{i\theta_{D_0}(u)} = e^{i\theta_D(u)} + O(n^{-1/4}).$$

Lemma. Let $0 < \epsilon < 1/6$ and let $0 < \rho < 1$ be fixed. If $x,\, y \in V_0$ with $|\psi(x)| \leq \rho$ and $|\psi(y)| \geq 1 - n^{-(1/6-\epsilon)}$, then

$$\frac{1 - |\psi_{D_0}(x)|^2}{|\psi_{D_0}(x) - e^{i\theta_{D_0}(y)}|^2} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - e^{i\theta_D(y)}|^2} + O(n^{-1/4}).$$

Therefore, if we know

$$\frac{H_{D_0}(x,u)}{H_{D_0}(0,u)} = \frac{1 - |\psi_{D_0}(x)|^2}{|\psi_{D_0}(x) - \psi_{D_0}(u)|^2} \cdot [\, 1 + O(n^{-(1/6-\epsilon)})\,],$$

then the lemma implies

$$\frac{H_D(x,u)}{H_D(0,u)} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - \psi_D(u)|^2} \cdot [\, 1 + O(n^{-(1/6-\epsilon)})\,].$$

Proving the Closeness of the Poisson Kernels

To prove

$$\frac{H_{D_0}(x,u)}{H_{D_0}(0,u)} = \frac{1-|\psi_{D_0}(x)|^2}{|\psi_{D_0}(x)-\psi_{D_0}(u)|^2} \cdot [\, 1 + O(n^{-(1/6-\epsilon)})\,],$$

one can replace H by G, the discrete Green's function, and prove that equality using the KMT approximation and some technical estimates based on the growth and distortion theorems to deal with points close to the boundary.

The following lemma is needed to establish the Green's function estimates and produces the exponent of $1/6$.

Lemma (K-Lawler, 2005; B-J-K, 2009).

$$\left| \mathbb{E}^x \left[\, \log |B_{T_D}| - \log |S_{\tau_D}|\, \right] \right| \le c n^{-1/3} \log n.$$

Note. Solving $p - 1/3 = -p$ implies $p = 1/6$.

Lemma? (B-J-K, 2010).

$$\left| \mathbb{E}^x \left[\, \log |B_{T_D}| - \log |S_{\tau_D}|\, \right] \right| \le c n^{-1/2} \log n.$$

Note. $p - 1/2 = -p$ implies $p = 1/4$ and this appears to be optimal.

Step 2. Finding the Mean and Variance of the Driving Function

We now apply

$$\frac{H_D(x,u)}{H_D(0,u)} = \frac{1-|\psi_D(x)|^2}{|\psi_D(x)-\psi_D(u)|^2} \cdot [\, 1 + O(n^{-(1/6-\epsilon)})\,],$$

to the domains $D_n \setminus \gamma_n[0,k]$:

Choose $k = k(n)$ so that t_k, the capacity of $\gamma_n[0,k]$ is on an intermediate scale (of order $n^{-1/18}$).

Fix an appropriate $z \in V(D_n)$ and set

$$\lambda_k := \frac{1-|\psi_k(z)|^2}{|\psi_k(z)-\psi_k(\gamma_n(k))|^2} = \mathrm{Re}\left(\frac{\psi_k(\gamma_n(k))+\psi_k(z)}{\psi_k(\gamma_n(k))-\psi_k(z)}\right).$$

This is almost a martingale with respect to $\gamma_n[0,j]$.

We can express λ_k in terms of ψ_{D_0}, the Loewner equation, t_k and $\vartheta(t_k) = \psi_k(\gamma_n(k))$.

Finding the Mean and Variance of the Driving Function

We can Taylor-expand $\lambda_k - \lambda_0$, take expectations, and compare coefficients; the fact that λ_k is almost a martingale implies (after some work involving the Beurling and distortion estimates) that

$$\mathbb{E}[\vartheta(t_k)] = O(n^{-(1/6-\epsilon)})$$

and

$$\mathbb{E}[\vartheta(t_k)^2 - 2t_k] = O(n^{-(1/6-\epsilon)}).$$

Step 3. Coupling the Driving Function with Brownian Motion

Grow a macroscopic piece of the curve (capacity of order 1), pieced together by $\approx n^{1/18}$ intermediate scale pieces of the curve ($\gamma([t_{m_{k-1}}, t_{m_k}])$).

The pieces correspond to increasing times/capacities t_{m_k}. From Step 2, $k \mapsto \vartheta(t_{m_k})$ is almost, though not quite, a martingale (with small increments).

However,

$$\xi_j = \vartheta(t_{m_j}) - \vartheta(t_{m_{j-1}}) - \mathbb{E}[\vartheta(t_{m_j}) - \vartheta(t_{m_{j-1}}) | \gamma[0, t_{m_{j-1}}]]$$

is a martingale difference sequence and

$$M_k = \sum_{j=0}^{k} \xi_j$$

can be embedded into Brownian motion.

Coupling the Driving Function with Brownian Motion

Lemma (Skorokhod embedding theorem). If $(M_n)_{n\le N}$ is an $(\mathcal{F}_n)_{n\le N}$ martingale, with $\|M_n - M_{n-1}\|_\infty \le 2\,\delta$ and $M_0 = 0$ a.s., then there are stopping times $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_N$ for standard Brownian motion $(B_t, t \ge 0)$, such that $(M_0, M_1, \ldots, M_N)$ and $(B(\tau_0), B(\tau_1), \ldots, B(\tau_N))$ have the same law. Moreover, one can impose for $n = 0, 1, \ldots, N-1$

$$\mathbb{E}\big[\tau_{n+1} - \tau_n \mid B[0,\tau_n]\big] = \mathbb{E}\big[(B(\tau_{n+1}) - B(\tau_n))^2 \mid B[0,\tau_n]\big],$$

$$\mathbb{E}\big[(\tau_{n+1} - \tau_n)^p \mid B[0,\tau_n]\big] \le C_p \mathbb{E}\big[(B(\tau_{n+1}) - B(\tau_n))^{2p} \mid B[0,\tau_n]\big]$$

for constants $C_p < \infty$, and also

$$\tau_{n+1} \le \inf\{t \ge \tau_n : |B_t - B_{\tau_n}| \ge 2\,\delta\}\,.$$

Coupling the Driving Function with Brownian Motion

We get stopping times τ_j such that

$$B(\tau_j) \approx \vartheta(t_{m_j}).$$

To show that the "Brownian motion time" is close to $2\times$ capacity time, i.e., $\tau_j \approx 2t_{m_j}$, we consider the natural "time" associated to M, namely

$$Y_k := \sum_{j=1}^{k} \xi_j^2, \quad k = 1, \ldots, K,$$

and first show that Y_k is close to $2t_{m_k}$, using a martingale maximal inequality due to Haeusler and the fact that

$$\mathbb{E}[\vartheta(t_k)^2 - 2t_k] \approx 0.$$

Haeusler's Inequality

Lemma (Haeusler). Let ξ_k, $k = 1, \ldots, K$, be a martingale difference sequence with respect to the filtration $\mathcal{F}_k$. Then for all $\lambda, u, v > 0$

$$\mathbf{P}\left(\max_{1\le j\le K} |\sum_{k=1}^{j} \xi_k| \ge \lambda\right) \le \sum_{k=1}^{K} \mathbf{P}(|\xi_k| > u) + 2\,\mathbf{P}\left(\sum_{k=1}^{K} \mathbb{E}(\xi_k^2|\mathcal{F}_{k-1}) > v\right) + \exp\{\lambda u^{-1}(1 - \log(\lambda u v^{-1}))\}.$$

Remark. L-S-W use Doob's maximal inequality, but in order to obtain a better rate of convergence we use Haeusler's inequality.

Our Rate

- Step 1. Error is $O(n^{-(1/6-\epsilon)})$.
- Step 2. Error is (step 1)$^{1/3} = O(n^{-(1/18-\epsilon)})$.
- Step 3. Error is (step 2)$^{1/2} = O(n^{-(1/36-\epsilon)})$.

Theorem (Beneš-Johansson-K, 2009). Let $0 < \epsilon < 1/36$ be fixed, and let D be a simply connected domain with $\mathrm{inrad}(D) = 1$. For every $T > 0$ there exists an $n_0 < \infty$ depending only on T such that whenever $n > n_0$ there is a coupling of γ_n with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that

$$\mathbf{P}\left(\sup_{0\leq t\leq T} |W_n(t) - e^{iB(2t)}| > n^{-(1/36-\epsilon)}\right) < n^{-(1/36-\epsilon)}.$$

Recall that

$$W_n(t) = W_n(0)e^{i\vartheta_n(t)}, \quad t \geq 0,$$

denotes the Loewner driving function for the curve $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ parameterized by capacity.

Final Comments

- The overall rate of convergence depends on the rates obtained in the 3 steps. The rate in the last step is "universal" while the rate in the first two steps depends on the martingale observable (m.g.o.).
- The rate in the first step can probably be improved to $1/4$. The rates in the last two steps probably cannot be improved. Thus, the best our technique could give is $O(n^{-(1/24-\epsilon)})$.
- Even though we have a rate based on this m.g.o. (perhaps even the optimal rate for this m.g.o.), we can still ask about a better/optimal rate.
- It seems that we might be able to obtain a non-trivial rate of convergence in the Hausdorff metric.
- Future work? Obtaining a rate for pathwise convergence in the stronger topology will be very difficult. At the moment, we cannot transfer a rate for the driving processes to a rate for the paths.