A random look at loop-erased walk and the Schramm-Loewner evolution

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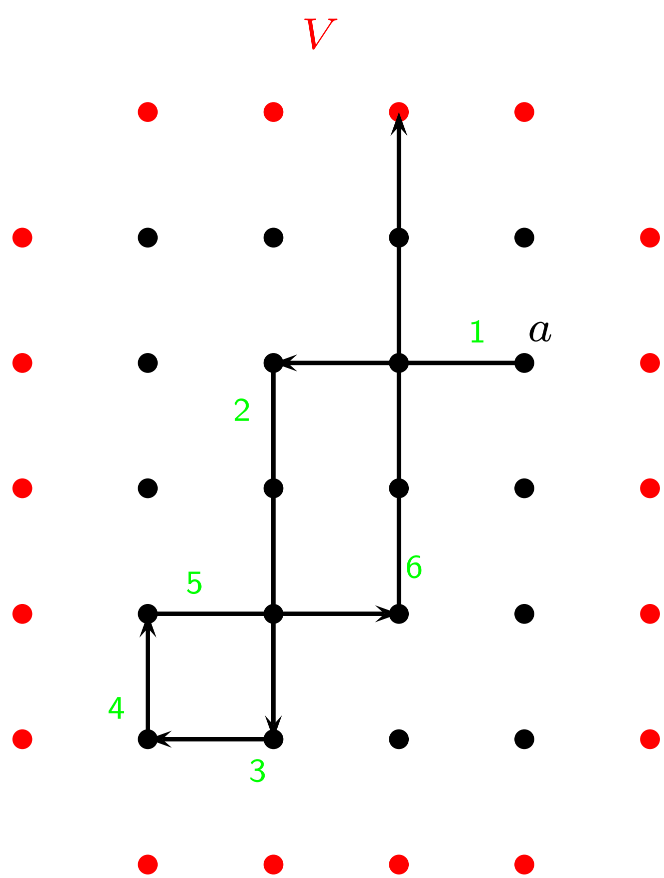
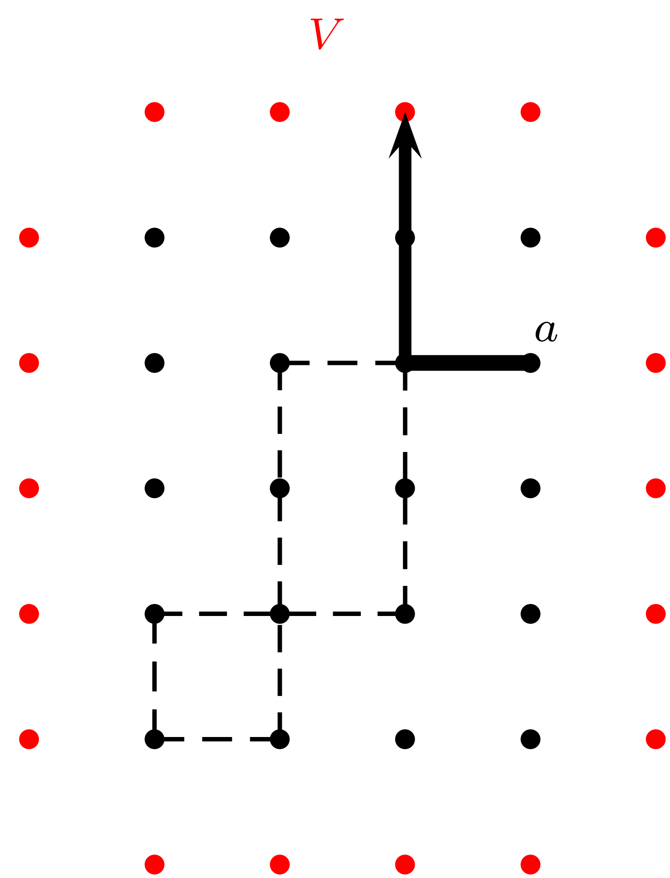
Brief History

- The self-avoiding walk is a model of polymer chains introduced in 1949 by the Nobel Prize-winning chemist Paul Flory.
- The loop-erased random walk is a model introduced in 1980 by Greg Lawler in an attempt to understand the self-avoiding walk.
- LERW is obtained by chronologically erasing loops of a random walk path to produce a path with no self-intersections.
- It was eventually proved to be in a different universality class than the SAW, but is an interesting model in its own right.
- In the mid-90s, David Wilson and others showed how LERW was intimately connected with the generation of uniform spanning trees.
- In high dimensions, simple random walks are transient and tend not to self-intersect. Since random walk converges to Brownian motion, it is plausible that SAW converges to BM and LERW converges to BM in high dimensions.
- In 2002, Oded Schramm discovered the Schamm-Loewner evolution (SLE) while attempting to understand the scaling limit of two-dimensional LERW.

Loop-Erased Random Walk

Consider a connected graph $G \subsetneq \mathbb{Z}^2$, a vertex $a \in G$, and a nonempty set $V \subset G$. Loop-erased random walk (LERW) γ from a to V is defined as follows.





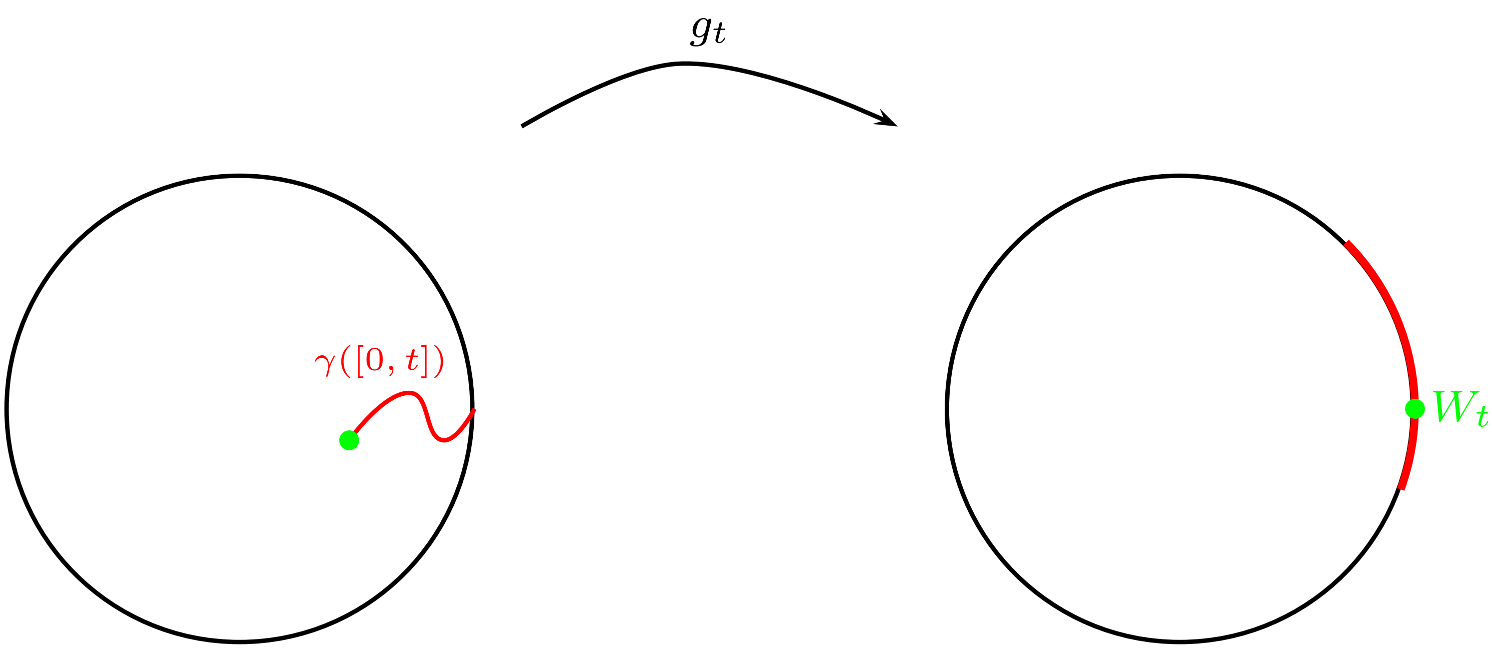
Loop-Erased Random Walk

Let $\{S_n\}_{n\geq 0}$ be simple random walk on G and $\tau_V = \inf\{n \geq 0 : S_n \in V\}$.
$\gamma = (\gamma_0, \ldots, \gamma_\ell)$ is defined inductively by

- $\gamma_0 = S_0 = a$,
- for $n \geq 0$,
  - if $\gamma_n \in V$, then $n = \ell$,
  - if $\gamma_n \notin V$, then $\gamma_{n+1} = S_k$, where $k = \max\{m \leq \tau_V : S_m = \gamma_n\}$.

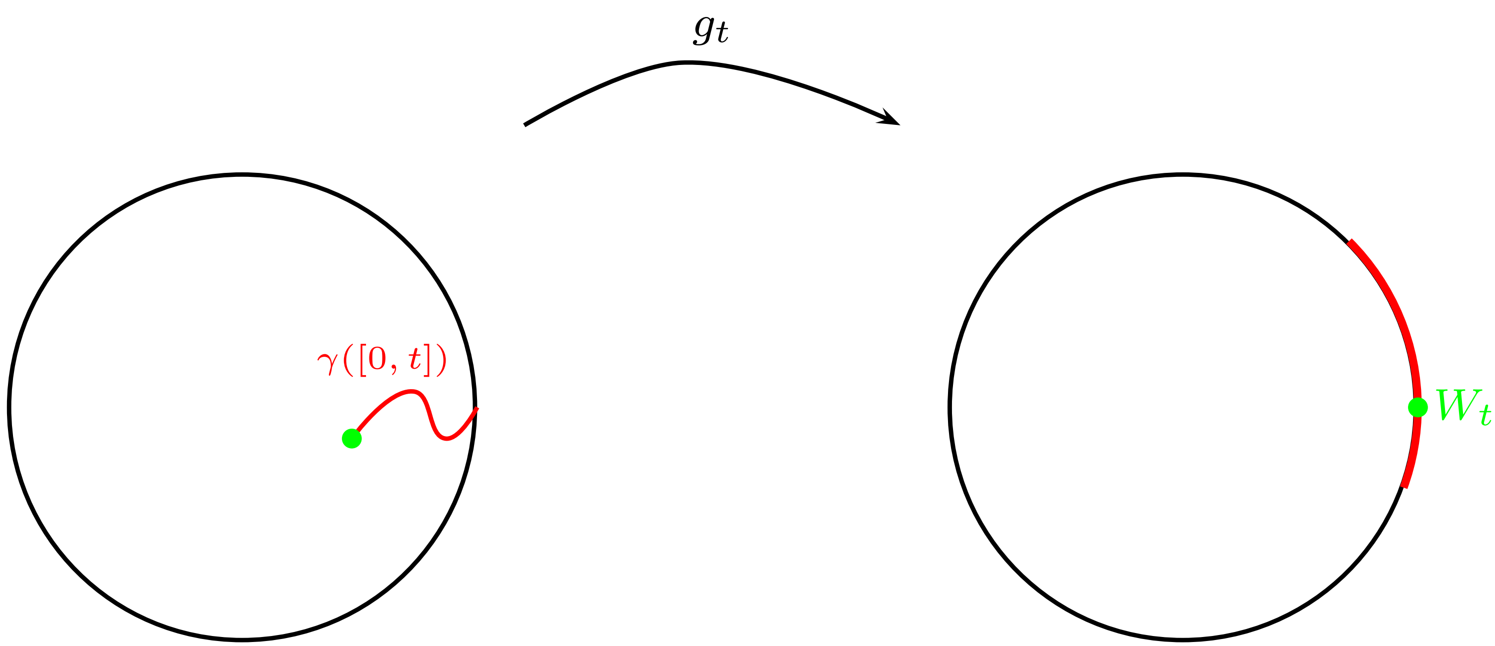
The loop-erasure of S from a to V and of its time-reversal are not usually the same (path by path). However, they have the same distribution.

Radial SLE



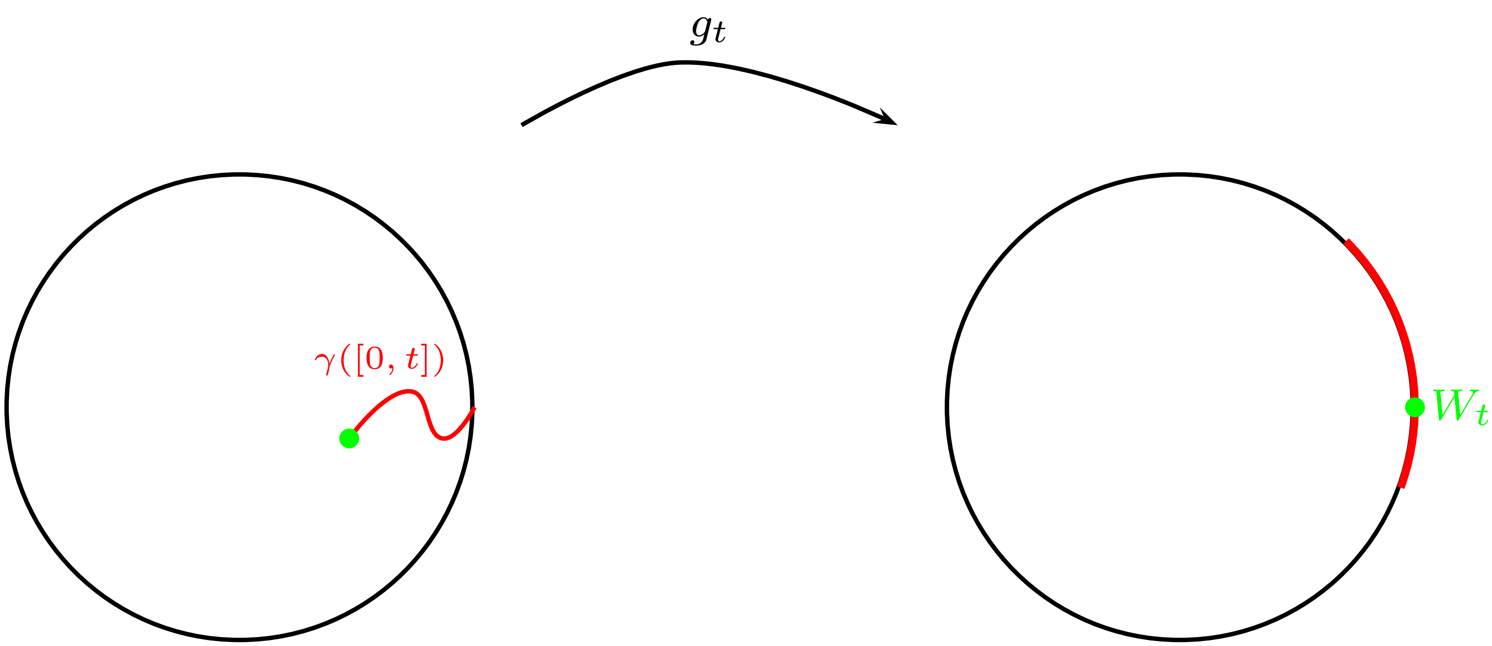
Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk, and consider a simple (non-self-intersecting) curve $\gamma : [0,\infty] \to \overline{\mathbb{D}}$ with $\gamma(0) = 1 \in \partial\mathbb{D}$, $\gamma(\infty) = 0$, and $\gamma(0,\infty) \subset \mathbb{D}$.

Radial SLE



For every fixed $t \geq 0$, the slit disk $\mathbb{D}_t := \mathbb{D} \setminus \gamma(0,t]$ is simply connected and so by the Riemann mapping theorem, there exists a unique conformal transformation $g_t : \mathbb{D}_t \to \mathbb{D}$ satisfying $g_t(0) = 0$ and $g_t'(0) > 0$.

Radial SLE

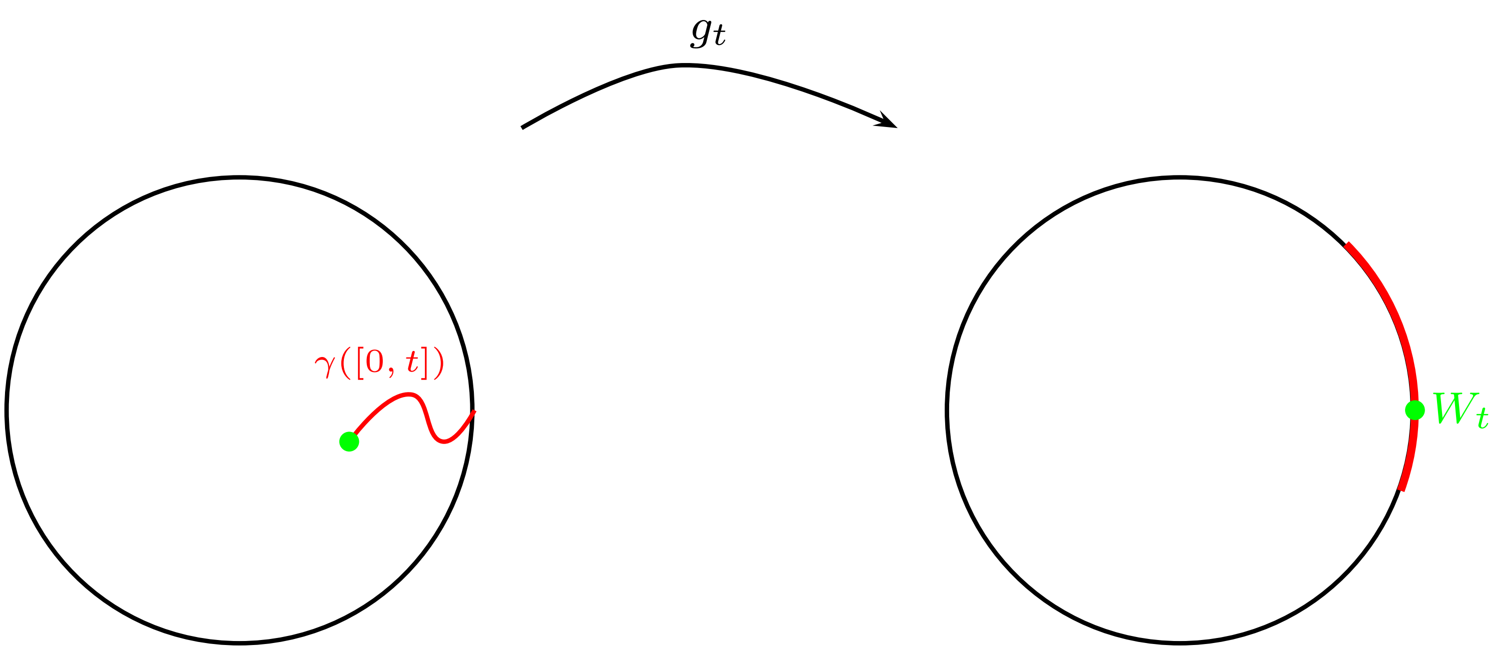


The function $t \mapsto g_t'$ is increasing so we can reparametrize γ so that

$$g_t'(0) = e^t.$$

This is the capacity parametrization.

Radial SLE



It can be shown that there is a unique point $W_t \in \partial\mathbb{D}$ for all $t \geq 0$ with $W_t := g_t(\gamma(t))$ and that the function $t \mapsto W_t$ is continuous.

Radial SLE

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{D}_t \to \mathbb{D}$, can be described by a PDE involving W_t known as the Loewner equation.

For $z \in \mathbb{D}$ with $z \notin \gamma[0, \infty]$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy

$$\frac{\partial}{\partial t}\, g_t(z) = g_t(z)\frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z,$$

where

$$W_t = \lim_{z \to \gamma(t)} g_t(z).$$

We call W the <u>driving function</u> of the curve γ.

Radial SLE

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z. \qquad (*)$$

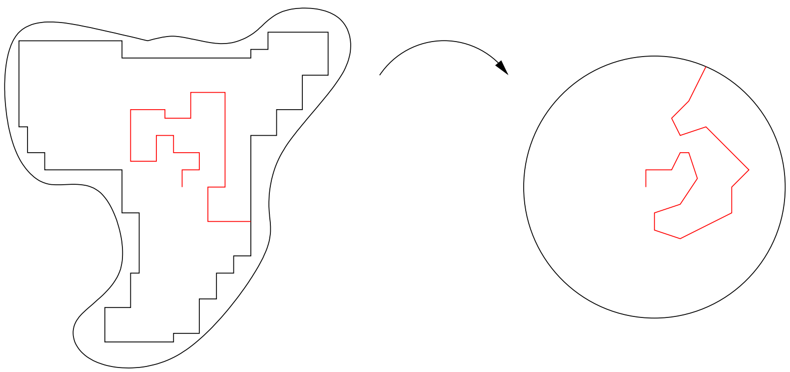
The obvious thing to do now is to start with a continuous function $t \mapsto W_t$ from $[0, \infty)$ to $\partial\mathbb{D}$ and solve the Loewner equation for g_t.

Ideally, we would like to solve $(*)$ for g_t, define simple curves $\gamma(t)$, $t \geq 0$, by setting $\gamma(t) = g_t^{-1}(W_t)$, and have g_t map $\mathbb{D} \setminus \gamma(0, t]$ conformally onto $\mathbb{D}$.

Although this is the intuition, it is not quite precise because we see from the denominator on the right-side of $(*)$ that problems can occur if $W_t - g_t(z) = 0$.

Formally, if we let T_z be the supremum of all t such that the solution to $(*)$ is well-defined up to time t with $g_t(z) \in \mathbb{D}$, and we define $\mathbb{D}_t = \{z : T_z > t\}$, then g_t is the unique conformal transformation of $\mathbb{D}_t$ onto $\mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$.

Radial SLE: Oded Schramm's idea



Assuming that the scaling limit of LERW exists, is conformally invariant, and satisfies the domain Markov property, then Brownian motion is the only possibility for the driving function, say

$$W_t = e^{i\sqrt{\kappa}B_t}$$

where B_t is a one-dimensional Brownian motion starting at 0 with variance parameter $\kappa \geq 0$.

Definition of Radial SLE

The **radial Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard (or capacity) parametrization** (or simply SLE_κ) is the random collection of conformal maps $\{g_t,\, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t}\, g_t(z) = g_t(z)\,\frac{e^{i\sqrt{\kappa}B_t} + g_t(z)}{e^{i\sqrt{\kappa}B_t} - g_t(z)}, \quad g_0(z) = z. \tag{LE}$$

where B_t is a standard one-dimensional Brownian motion.

Review of Radial SLE (cont)

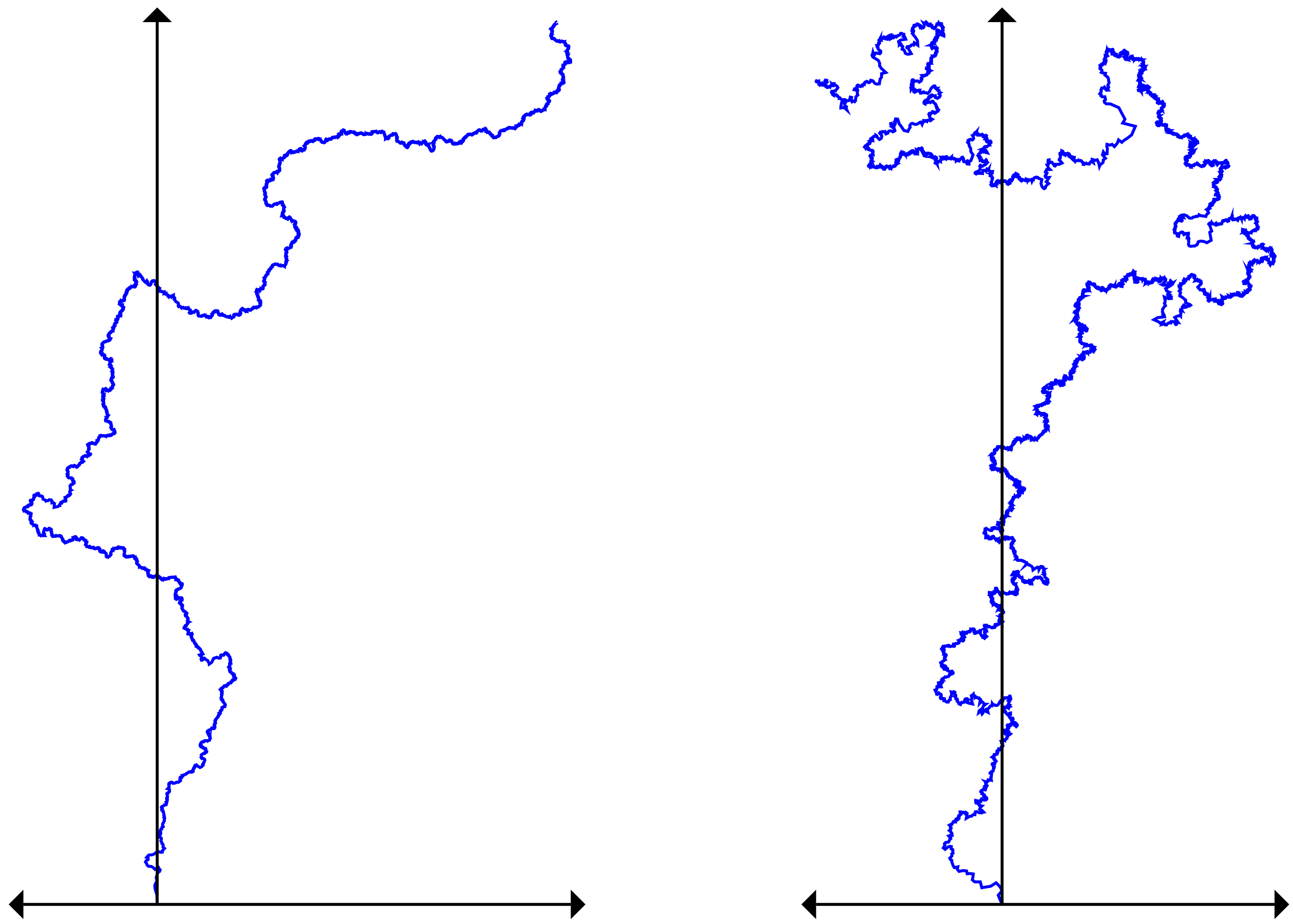
The question is now whether there exists a curve associated with the maps g_t. (Rohde-Schramm, Lawler-Schramm-Werner)

- If $0 < \kappa \le 4$, then there exists a random simple curve $\gamma : [0, \infty] \to \overline{\mathbb{D}}$ given by $\gamma(t) = g_t^{-1}(e^{i\sqrt{\kappa}B_t})$. For this range of κ, our intuition matches the theory!

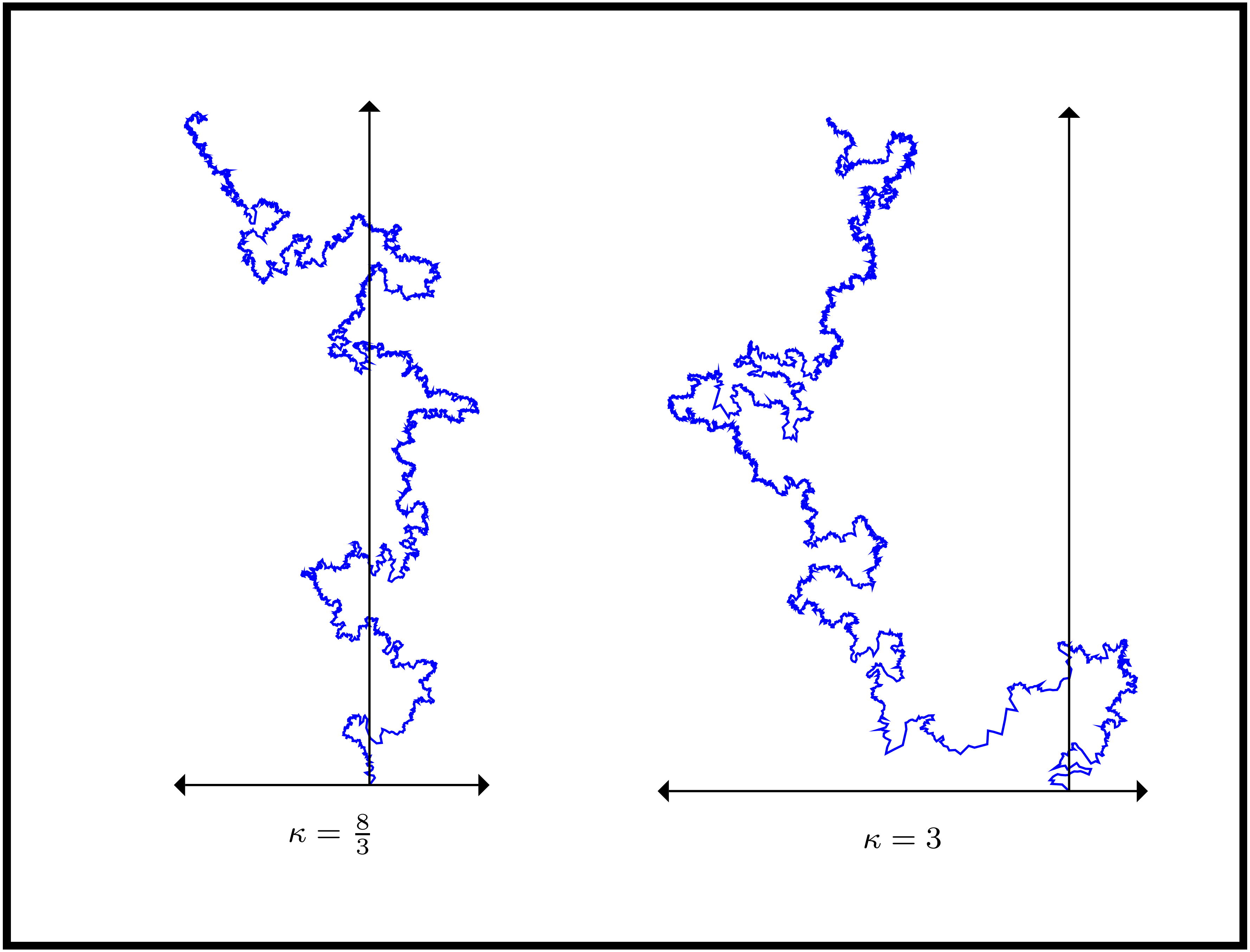
- For $4 < \kappa < 8$, there exists a random curve $\gamma : [0, \infty] \to \overline{\mathbb{D}}$. These curves have double points and they hit $\partial\mathbb{D}$, but they never cross themselves! The maps g_t are conformal transformations of $\mathbb{D}_t$ onto $\mathbb{D}$. We think of $\mathbb{D}_t = \mathbb{D} \setminus K_t$ where K_t is the hull of $\gamma(0, t]$ visualized by taking $\gamma(0, t]$ and filling in the holes.

- For $\kappa \ge 8$, there exists a random curve $\gamma : [0, \infty] \to \overline{\mathbb{D}}$ which is space-filling! Furthermore, it has double points, but does not cross itself!

As a result, we also refer to the curve γ as radial SLE$_\kappa$. SLE paths are extremely rough: the Hausdorff dimension of a radial SLE$_\kappa$ path is $\min\{1 + \kappa/8, 2\}$ (Beffara).



$\kappa = 1$ $\kappa = 2$



$\kappa = \frac{8}{3}$
$\kappa = 3$

Proving Convergence Result with SLE

The following refined martingale principle is essentially due to Smirnov and is one way to prove convergence to SLE.

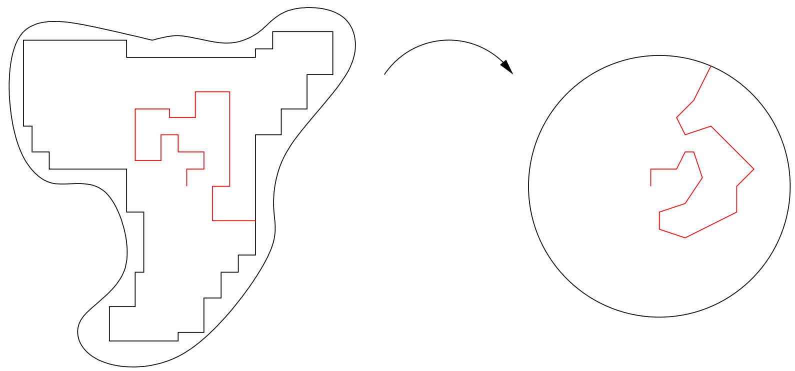
Martingale Observable Principle. If γ is a random curve that admits a non-trivial conformal martingale

$$M_t(z) = M(z; \gamma(t), D, a, b),$$

then γ is given by SLE with parameter κ derived from M_t.

- For each model, one needs to find a suitable martingale observable.
- At the moment, there is no general "template" for constructing such martingales.
- Even if one has a martingale observable, there are still technical questions particular to each model about what to do with it and how to actually carry through the proof of convergence.
- In many cases, the martingale observable at time $t \to \infty$ reduces to the probability of a particular event. Computing that probability is often a key step in the proofs of convergence.

From LERW to SLE



- Let $D \ni 0$ be a simply connected planar domain with $\frac{1}{n}\mathbb{Z}^2$ grid domain approximation $D_n \subset \mathbb{C}$. A grid domain is a domain whose boundary is a union of edges of the scaled lattice. That is, D_n is the connected component containing 0 in the complement of the closed faces of $n^{-1}\mathbb{Z}^2$ intersecting ∂D. Note that D_n is simply connected.
- $\psi_{D_n} : D_n \to \mathbb{D}$, $\psi_{D_n}(0) = 0, \psi'_{D_n}(0) > 0$.
- γ_n: time-reversed LERW from 0 to ∂D_n (on $\frac{1}{n}\mathbb{Z}^2$).
- $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ is a path in $\mathbb{D}$. Parameterize by capacity.
- $W_n(t) = W_0 e^{i\vartheta_n(t)}$: the Loewner driving function for $\tilde{\gamma}_n$.

L-S-W Prove Loop-Erased Random Walk Converges to SLE(2)

Theorem (Lawler-Schramm-Werner, 2004). Let $\mathcal{D}$ be the set of simply connected grid domains with $0 \in D, D \neq \mathbb{C}$. For every $T > 0, \varepsilon > 0$, there exists $n = n(T, \varepsilon)$ such that if $D \in \mathcal{D}$ with inrad$(D) > n$, then there exists a coupling between loop-erased random walk γ from ∂D to 0 in D and Brownian motion B started uniformly on $[0, 2\pi]$ such that

$$\mathbf{P}\left\{ \sup_{0 \leq t \leq T} |\theta(t) - B(2t)| > \varepsilon \right\} < \varepsilon,$$

where $\theta(t)$ satisfies $W(t) = W(0)e^{i\theta(t)}$ and $W(t)$ is the driving process of γ in Loewner's equation.

This is "a kind of of convergence" of LERW to SLE_2, and leads (without too much difficulty) to the stronger convergence (which we won't discuss) of paths with respect to the Hausdorff metric.

L-S-W then use this result to establish convergence of paths with respect to the metric that identifies curves modulo reparametrization.

L-S-W Prove Loop-Erased Random Walk Converges to SLE(2)

Consider the following metric on the space of curves in $\mathbb{C}$:

$$\rho(\gamma_1, \gamma_2) = \inf_{\phi} \sup_{0 \le t \le 1} |\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)|$$

where the infimum is over all choices of parametrizations $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $[0, 1]$ of γ_1 and γ_2.

Let μ_n denote the law of γ_n, time-reversed LERW from 0 to ∂D_n, and let μ denote the law of the image in D of radial SLE(2).

Theorem (Lawler-Schramm-Werner, 2004).
The measures μ_n converge weakly to μ as $n \to \infty$ with respect to the metric ρ on the space of curves.

A Rate of Convergence for the Driving Process

Recall that

$$W_n(t) = W_n(0)e^{i\vartheta_n(t)}, \quad t \geq 0,$$

denotes the Loewner driving function for the curve $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ parameterized by capacity.

Theorem (Beneš-Viklund-K, 2013). Let $0 < \epsilon < 1/24$ be fixed, and let D be a simply connected domain with inrad$(D) = 1$. For every $T > 0$ there exists an $n_0 < \infty$ depending only on T such that whenever $n > n_0$ there is a coupling of γ_n with Brownian motion $B(t)$, $t \geq 0$, where $e^{iB(0)}$ is uniformly distributed on the unit circle, with the property that

$$\mathbf{P}\left(\sup_{0 \leq t \leq T} |W_n(t) - e^{iB(2t)}| > n^{-(1/24-\epsilon)} \right) < n^{-(1/24-\epsilon)}.$$

This leads to a rate of convergence of the paths with respect to the Hausdorff metric.

Ideas of Proof

L-S-W follow these three main steps to prove convergence of the driving processes.

1. Find a discrete martingale observable for the LERW path. Prove that it converges to something conformally invariant.
2. Use Step 1 together with the Loewner equation to show that the Loewner driving function for the LERW is almost a martingale with "correct" (conditional) variance.
3. Use Step 2 and Skorokhod embedding to couple the Loewner driving function for the LERW with a Brownian motion and show that they are uniformly close with high probability.

To obtain a rate we have to re-examine the steps to find explicit bounds on error terms.

A Rate of Convergence for the Paths

Theorem (Viklund, 2014). For each n sufficiently large, there is a coupling of $\tilde{\gamma}_n$ with $\tilde{\gamma}$ such that

$$\mathbf{P}\left(\sup_{t\in[0,\sigma]} |\tilde{\gamma}_n - \tilde{\gamma}| > \epsilon_n^{1/41}\right) < \epsilon_n^{1/41}.$$

where both curves are parameterized by capacity, $\epsilon_n = n^{-1/24}$ is the convergence rate of the driving terms and σ is a stopping time. The same estimate holds for the preimages of the curves in D_n.

Important. These theorems all tell us that the LERW and SLE(2) traces are close. They do not tell us that they are close in space at roughly the same time.

We would like to prove that LERW also converges to SLE(2) in the so-called natural parametrization.

Motivation: Random Walk Converges to Brownian Motion

$$\left\{t \mapsto \frac{1}{n} S(n^2 t \wedge \tau_n)\right\} \xrightarrow{(d)} \{t \mapsto B(t \wedge \tau_1)\}$$

S – simple random walk on $\mathbb{Z}^2$ with $S_0 = 0$

B – complex Brownian motion with $B_0 = 0$

τ_r – first time curve hits the circle of radius r

Convergence in the strong topology

$$d(\gamma_1, \gamma_2) = |t_{\gamma_1} - t_{\gamma_2}| + \sup_{0 \le t \le t_{\gamma_1} \vee t_{\gamma_2}} |\gamma_1(t) - \gamma_2(t)|$$

where t_γ is the lifetime of the curve γ.

– i.e., weak convergence of probability measures on metric space of curves

– accounts for different random curves running for different lengths of time

Motivation: Random Walk Converges to Brownian Motion

$$\left\{t \mapsto \frac{1}{n} S(n^2 t \wedge \tau_n)\right\} \xrightarrow{(d)} \{t \mapsto B(t \wedge \tau_1)\}$$

We want convergence of random walk to Brownian motion stopped when it exits the unit disk $\mathbb{D}$. We know (functional CLT) that we need to scale space by the square root of time. It is notationally easier if we scale space by $1/n$; that is, we approximate the disk by

$$\frac{1}{n}\mathbb{Z}^2 \cap \mathbb{D}$$

and so we can equivalently consider random walks on $\mathbb{Z}^2 \cap n\mathbb{D}$. Note that $n^2 \leq \mathbb{E}[\tau_n] \leq (n+1)^2$; we expect the random walk to take $\sim n^2$ steps to exit the ball of radius n. Thus, in order to associate the "correct" continuous curve to the random walk path, we need to introduce the speed function $\sigma_n(t) = \mathbb{E}[\tau_n]t$ or $\sigma_n(t) = n^2 t$.

Important. Not only are the random walk and the Brownian motion traces close, they are close in space at roughly the same time.

Convergence of LERW to SLE(2) in the Natural Parametrization

Suppose that X is a LERW on $\mathbb{Z}^2$ started at the origin. We would like

(i) to show that there is a speed function $t \mapsto \sigma_n(t)$ so that

$$t \mapsto \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

converges in law under the strong topology, and

(ii) to identify the limiting curve as SLE(2) in the natural time parametrization that was recently introduced by Lawler-Sheffield and Lawler-Zhou.

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Most desirable choice is $\sigma_n(t) = n^{5/4} t$

Based on the long-standing conjecture that M_n "grows like" $n^{5/4}$ where M_n is the number of steps in the LERW (i.e., $M_n = \tau_n$)

Very, very difficult to prove! This would imply that

$$\frac{M_n}{n^{5/4}}$$

has a limiting distribution as $n \to \infty$.

Strongest known result is still that

$$\lim_{n\to\infty} \frac{\log M_n}{\log n} = \frac{5}{4}.$$

(Originally proved by Kenyon, later by Masson.)

But we don't even know how to get tightness!

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Second choice is $\sigma_n(t) = \mathbb{E}[M_n]t$

This implies that the total lifetime of Y_n is $M_n/\mathbb{E}[M_n]$

Barlow and Masson give tightness bounds for this. In fact, they also give exponential tail bounds

$$\mathbf{P}\left\{\alpha^{-1} \leq \frac{M_n}{\mathbb{E}[M_n]} \leq \alpha\right\} \geq 1 - Ce^{-c\alpha^{1/2}}.$$

Another advantage: If this works, then showing convergence for the first choice of speed function reduces to showing that

$$\mathbb{E}[M_n] \sim cn^{5/4}.$$

An Occupation Measure

If γ is a curve, then its occupation measure ν_γ identifies the amount of time γ spends in each Borel subset of $\mathbb{C}$.

Formally,

$$\nu_\gamma(A) := \int_0^{t_\gamma} 1\{\gamma(s) \in A\}\, \mathrm{d}s$$

where A is a Borel subset of $\mathbb{C}$.

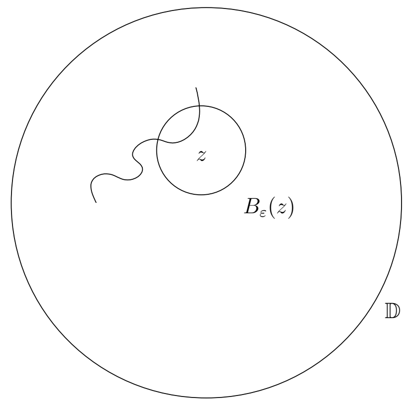
Note. Implicit in the statement that γ is a curve is its time parametrization.

- ν_γ is supported on γ
- The total mass of ν_γ is t_γ

Key observation.

occupation measure + curve modulo reparametrization $\Rightarrow$ original curve

How should the speed function by chosen?



Would like to prove that the curve up until it hits the ball of radius ϵ does not too strongly affect how the curve behaves inside the ball of radius ϵ.

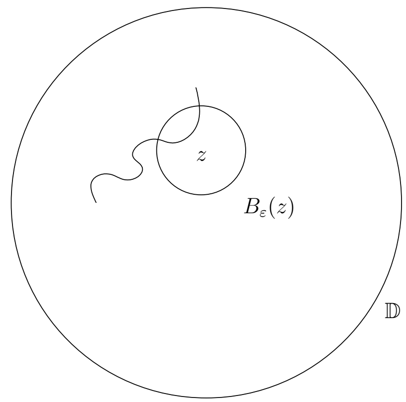
Conjecture. If $z \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small, then

$$\mathbb{E}\left[\nu_{Y_n}(B_\epsilon(z)) \mid Y_n \cap B_\epsilon(z) \neq \emptyset\right] = \frac{\mathbb{E}[M_{\epsilon n}]}{\mathbb{E}[M_n]} + o(1)$$

as $n \to \infty$.

To prove the conjecture, need a strong separation lemma. This is currently out of reach.

How should the speed function by chosen?



Theorem (Alberts-Masson-K, 2013). If $z \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small, then

$$\mathbb{E}\left[\, \nu_{Y_n}(B_\epsilon(z)) \mid Y_n \cap B_\epsilon(z) \neq \emptyset \right] \leq C \log(1/\epsilon)\epsilon^{5/4}.$$

How should the speed function by chosen?

Write $\sigma_n(t) = c_n t$.

It is sufficient to prove that

$$\sum_{e \in A_n} \left[\frac{2n^2}{c_n} \mathbb{P}\left(z_e \in \tilde{Y}_n \right) - G(z_e) \right] = o(n^2)$$

where $A_n = A \cap n^{-1}\mathbb{Z}^2$ so that the sum is over all (undirected) edges e of A_n, and z_e is the midpoint of the edge e.

Carrying out this estimate appears to be genuinely difficult. It is a hard problem to describe asymptotics for the probability that loop-erased walk passes through a particular edge, and even harder to show that the limit is the SLE Green's function. Recently, G. Lawler, C. Beneš, and F. Viklund studied the scaling limit of the LERW Green's function, although it is not clear if they can extend their results to get the sharp asymptotics needed for convergence of LERW to SLE(2) in the natural parametrization.

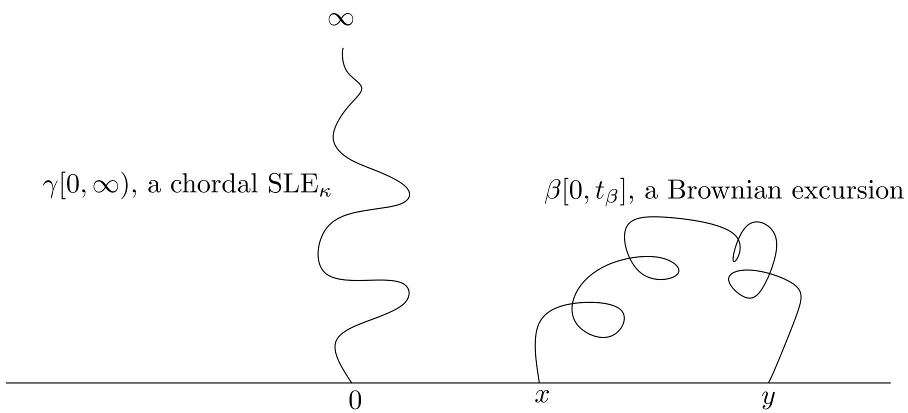
Areas of Active Research

SLE describes the scaling limit of a single interface. What about multiple interfaces?

This has been considered from a physical point of view by Bauer, Bernard, and Kytölä (2005). Mathematical approaches have been considered by Dubédat (2006) and by K and Lawler (2006).

Proving convergence for multiple interfaces in discrete models to multiple SLE is still an open problem for $\kappa \neq 3$. Izyurov (2013) proved multiple interfaces in the Ising model converge to multiple SLE(3).

Convergence of Multiple LERW to Multiple SLE$_2$



Theorem (K, 2009). Suppose that $0 < x < y < \infty$ are real numbers and let $\beta : [0, t_\beta] \to \overline{\mathbb{H}}$ be a Brownian excursion from x to y in $\mathbb{H}$. If $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a chordal SLE$_\kappa$, $0 < \kappa \le 4$, from 0 to ∞ in $\mathbb{H}$, then

$$\mathbf{P}\{\,\gamma[0,\infty) \cap \beta[0,t_\beta] = \emptyset\,\} = \frac{\Gamma(2a)\Gamma(4a+1)}{\Gamma(2a+2)\Gamma(4a-1)}\,(x/y)\,F(2, 1-2a, 2a+2; x/y)$$

where $F = {}_2F_1$ is the hypergeometric function and $a = 2/\kappa$.