Bayes' rule for quantum random variables and positive operator valued measures

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References

(F-P-S) *Classical and nonclassical randomness in quantum measurements* by Douglas Farenick, Sarah Plosker, and Jerrod Smith. *J. Math. Phys.*, 52:122204, 2011.

(F-K) *Conditional expectation and Bayes rule for quantum random variables and positive operator valued measures* by Douglas Farenick and MJK. *J. Math. Phys.*, 53:042201, 2012.

Background

A measurement of a quantum system is represented mathematically by a positive operator valued measure ν which is defined on a σ-algebra of measurement events such that whenever a measurement is made with the system in state ρ, the measurement event E will occur with probability

$$\mathrm{Tr}(\rho\nu(E)).$$

Reference. *The Quantum Theory of Measurement* by Busch, Lahti, Mittelstaedt, LNP, Springer, 1991.

In practice, quantum measurements of an actual physical system are made by way of some apparatus and so X is often assumed to by finite.

Mathematically, however, there is no need for such a restriction and so one of our goals is to approach the theory of quantum measurement under the assumption that X be arbitrary.

Some notation

X, a locally compact Hausdorff space

$\mathcal{O}(X)$, the Borel σ-algebra of subsets of X

$\mathcal{F}(X)$, a sub-σ-algebra of $\mathcal{O}(X)$

$\mathcal{H}$, a d-dimensional Hilbert space

$\mathcal{B}(\mathcal{H})$, the space of (bounded) linear operators on $\mathcal{H}$

$\mathrm{Tr} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, the canonical trace functional

$\mathcal{B}(\mathcal{H})_+ = \{a \in \mathcal{B}(\mathcal{H}) : \langle a\zeta, \zeta\rangle \geq 0 \ \ \forall \ \ \zeta \in \mathcal{H}\}$, the space of positive operators

Positive operator valued measures

A set function $\nu : \mathcal{F}(X) \to \mathcal{B}(\mathcal{H})$ is called a positive operator valued measure on $(X, \mathcal{F}(X))$ if

1. $\nu(E)$ is a quantum effect for every $E \in \mathcal{F}(X)$, i.e., $\nu(E)$ is a positive operator with eigenvalues in $[0, 1]$,
2. $\nu(X) \neq 0$, and
3. for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}(X)$ with $E_j \cap E_k = \emptyset$ for $j \neq k$ we have
$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)$$
where the convergence on the right side of the previous equality is with respect to the σ-weak topology of $\mathcal{B}(\mathcal{H})$.

If $\nu(X) = 1 \in \mathcal{B}(\mathcal{H})$, we call it a positive operator valued probability measure.

Notation. $\mathrm{POVM}_{\mathcal{H}}(X)$ or $\mathrm{POVM}^1_{\mathcal{H}}(X)$

Quantum random variables

$\mathrm{S}(\mathcal{H})$, the state space of $\mathcal{H}$, is the set of all density operators $\rho \in \mathcal{B}(\mathcal{H})_+$ with $\mathrm{Tr}(\rho) = 1$.

Ex. If $\dim \mathcal{H} = d$, then $\rho = \frac{1}{d} 1 \in \mathrm{S}(\mathcal{H})$.

A quantum random variable on X is a function $\psi : X \to \mathcal{B}(\mathcal{H})$ such that

$$x \mapsto \mathrm{Tr}(\rho \psi(x))$$

is a complex random variable on X for every density operator $\rho \in \mathrm{S}(\mathcal{H})$.

Quantum averaging (F-P-S, F-K)

Theorem. There is a definition of integral whereby a quantum random variable ψ may be integrated against the positive operator valued probability measure ν to produce an operator

$$\mathbb{E}_\nu [\psi] := \int_X \psi \, \mathrm{d}\nu \in \mathcal{B}(\mathcal{H}).$$

Example. If $X = \{x_1, x_2, \ldots, x_n\}$, then

$$\mathbb{E}_\nu [\psi] = \sum_{j=1}^{n} h_j^{1/2} \psi(x_j) h_j^{1/2}$$

where $h_j = \nu(x_j)$.

The principal Radon-Nikodým derivative (F-P-S)

Let $\nu \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$ so that $\mu(E) = \frac{1}{d}\,\mathrm{Tr}(\nu(E))$ is a Borel measure.

Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathcal{H}$.

Let $\nu_{ij} : \mathcal{F}(X) \to \mathbb{C}$ be defined by $\nu_{ij}(E) = \langle \nu(E)e_j, e_i\rangle$ so that $\nu_{ij} \ll_{\mathrm{ac}} \mu$. By classical R-N Theorem, there exists a unique function

$$\frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu} \in L^1(X, \mathcal{F}(X), \mu)$$

such that

$$\nu_{ij}(E) = \int_E \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu}\,\mathrm{d}\mu.$$

The function

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \sum_{i,j=1}^{d} \frac{\mathrm{d}\nu_{ij}}{\mathrm{d}\mu} \otimes e_{ij}$$

where $e_{ij} \in \mathcal{B}(\mathcal{H})$ sends e_j to e_i and e_k to 0 is called the principal Radon-Nikodým derivative of ν.

Quantum averaging (F-P-S, F-K)

$\nu \in \mathrm{POVM}_X(\mathcal{H})$ and $\mu(E) = \frac{1}{d}\,\mathrm{Tr}(\nu(E))$

$I(\psi;\nu) = \int_X \psi \,\mathrm{d}\nu : \mathcal{H} \to \mathcal{H}$ is the unique operator having the property that

$$\mathrm{Tr}\left(\rho \int_X \psi \,\mathrm{d}\nu\right) = \int_X \mathrm{Tr}\left(\rho \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2} \psi(x) \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x)\right)^{1/2}\right) \mathrm{d}\mu$$

for every $\rho \in \mathrm{S}(X)$.

Example. If $X = \{x_1, x_2, \ldots, x_n\}$, then

$$\mathbb{E}_\nu\left[\psi\right] = \sum_{j=1}^{n} h_j^{1/2}\psi(x_j)h_j^{1/2}$$

where $h_j = \nu(x_j)$.

The non-principal Radon-Nikodým derivative (F-P-S)

Theorem. If $\nu_1, \nu_2 \in \mathrm{POVM}_{\mathcal{H}}(X, \mathcal{F}(X))$, then the following statements are equivalent.

1. $\nu_2 \ll_{\mathrm{ac}} \nu_1$, i.e., $\nu_2(E) = 0$ whenever $\nu_1(E) = 0$.
2. There exists a bounded ν_1-integrable $\mathcal{F}(X)$-measurable function $\varphi : (X, \mathcal{F}(X)) \to \mathcal{B}(\mathcal{H})$, unique up to sets of ν_1-measure zero, such that

$$\nu_2(E) = \int_E \varphi \, \mathrm{d}\nu_1 \tag{1}$$

   for every $E \in \mathcal{F}(X)$.

Moreover, if the equivalent conditions above hold and if $\mu_j = \mu_{\nu_j}$ is the finite Borel measure induced by ν_j, then $\mu_2 \ll_{\mathrm{ac}} \mu_1$ and

$$\varphi = \left(\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}\right) \left[\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1/2} \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\right) \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1/2} \right] \tag{2}$$

i.e., φ is the non-principal Radon-Nikodým derivative of ν_2 wrt ν_1 so we write

$$\frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \varphi.$$

A non-commutative multiplication (F-K)

If $a, b \in \mathcal{B}(\mathcal{H})_+$ are both invertible, then the geometric mean of a and b is defined by setting

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}.$$

Definition. Suppose that ν_1, $\nu_2 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with $\nu_2 \ll_{\mathrm{ac}} \nu_1$, and let $\mu_j = \mu_{\nu_j}$ be the induced Borel probability measures. If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a quantum random variable, then

$$\psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \left(\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1} \# \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\right)\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2} \psi \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{1/2} \left(\left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\right)^{-1} \# \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\right).$$

Remark. In the commutative setting—and, in particular, in the classical case of $\mathcal{H} = \mathbb{C}$—the multiplication defined by $\boxtimes$ reduces to ordinary multiplication. That is, if $a, b \in \mathcal{B}(\mathcal{H})_+$ commute, then $a\#b = a^{1/2}b^{1/2} = b^{1/2}a^{1/2} = b\#a$. Thus, if ψ, $\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}$, and $\frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}$ are pairwise commuting, then

$$\psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \psi\frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\psi.$$

Change of quantum measurement (F-K)

Theorem. Suppose that ν_1, $\nu_2 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with $\nu_2 \ll_{\mathrm{ac}} \nu_1$, and let $\mu_j = \mu_{\nu_j}$ be the induced Borel probability measures. If $\psi : X \to \mathcal{B}(\mathcal{H})$ is a ν_2-integrable quantum random variable, then

$$\psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}$$

is a ν_1-integrable quantum random variable and

$$\mathbb{E}_{\nu_2}[\psi] = \mathbb{E}_{\nu_1}\left[\psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\right]$$

or, equivalently,

$$\int_X \psi \,\mathrm{d}\nu_2 = \int_X \psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\,\mathrm{d}\nu_1 .$$

Theorems for the Radon-Nikodým derivatives (F-K)

Theorem (Chain Rule). If $\nu_1, \nu_2, \nu_3 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with $\nu_1 \ll_{\mathrm{ac}} \nu_2 \ll_{\mathrm{ac}} \nu_3$, then

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_3} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_3}.$$

Corollary. If $\nu_1, \nu_2 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with $\nu_2 \ll_{\mathrm{ac}} \nu_1$ and $\nu_1 \ll_{\mathrm{ac}} \nu_2$, then

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} \boxtimes \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2} = 1.$$

Quantum conditional expectation (F-K)

Theorem. Suppose that $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and that $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a ν-integrable quantum random variable with $\mathbb{E}_\nu[\psi] \neq 0$. If $\mathcal{F}(X)$ is a sub-σ-algebra of $\mathcal{O}(X)$, then there exists a function $\varphi : X \to \mathcal{B}(\mathcal{H})$ such that

1. φ is $\mathcal{F}(X)$-measurable,
2. φ is ν-integrable, and
3. $\mathbb{E}_\nu[\psi\chi_E] = \mathbb{E}_\nu[\varphi\chi_E]$ for every $E \in \mathcal{F}(X)$.

Moreover, if $\tilde{\varphi}$ is any other ν-integrable $\mathcal{F}(X)$-measurable function satisfying $\mathbb{E}_\nu[\psi\chi_E] = \mathbb{E}_\nu[\tilde{\varphi}\chi_E]$ for every $E \in \mathcal{F}(X)$, then $\nu(\{x \in X : \varphi(x) \neq \tilde{\varphi}(x)\}) = 0$.

We write

$$\varphi = \mathbb{E}_\nu[\psi|\mathcal{F}(X)].$$

Quantum Bayes' rule (F-K)

Theorem. Let ν_1, $\nu_2 \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ with $\nu_2 \ll_{\mathrm{ac}} \nu_1$ and $\nu_1 \ll_{\mathrm{ac}} \nu_2$.

If $\psi : X \to \mathcal{B}(\mathcal{H})_+$ is a quantum random variable with $\mathbb{E}_{\nu_2}[\psi] \neq 0$ and $\mathcal{F}(X)$ is a sub-σ-algebra of $\mathcal{O}(X)$, then

$$\mathbb{E}_{\nu_2}[\psi|\mathcal{F}(X)] \boxtimes \mathbb{E}_{\nu_1}\left[\frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\middle|\mathcal{F}(X)\right] = \mathbb{E}_{\nu_1}\left[\psi \boxtimes \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}\middle|\mathcal{F}(X)\right].$$

Some operator algebra results (F-K)

The quantum expectation of a constant is **not** necessarily constant.

Theorem. The set of $z \in \mathcal{B}(\mathcal{H})$ with $\mathbb{E}_\nu [z] = z$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

That is, if $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and $z \in \mathcal{B}(\mathcal{H})$, then

$$\mathbb{E}_\nu [z] \in \mathrm{C}^*\mathrm{conv}\,\{z\}.$$

Moreover, the function $\mathcal{E}_\nu : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\mathcal{E}_\nu(z) = \int_X z\,\mathrm{d}\nu\,, \quad z \in \mathcal{B}(\mathcal{H}),$$

is a unital quantum channel (i.e., a trace-preserving completely positive linear map).

Some operator algebra results (F-K)

Theorem. If $\nu \in \mathrm{POVM}^1_{\mathcal{H}}(X)$ and

$$\mathcal{A}_\nu \;=\; \mathrm{Span}_{\mathbb{C}}\left\{\rho \in \mathrm{S}(\mathcal{H}) \,:\, \mathbb{E}_\nu\left[\rho\right] = \rho\right\},$$

then

1. $\mathcal{A}_\nu$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, and
2. (Ergodic Property) there exists a trace-preserving unital completely positive linear map $\mathfrak{E}_\nu : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfying $\mathfrak{E}_\nu \circ \mathfrak{E}_\nu = \mathfrak{E}_\nu$ and having range $\mathcal{A}_\nu$ such that

$$\lim_{N\to\infty} \frac{1}{N}\left(\mathcal{I} + \sum_{j=1}^{N-1} \underbrace{\mathcal{E}_v \circ \cdots \circ \mathcal{E}_\nu}_{j}\right) \;=\; \mathfrak{E}_\nu\,.$$