Some partial results on the convergence of loop-erased random walk to SLE(2) in the natural parametrization

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Introduction

The plan is to discuss a strategy for showing convergence of loop-erased random walk on the two-dimensional square lattice to SLE(2), in the supremum norm topology that takes the time parametrization of the curves into account.

Very often we hear statements like the following.

- Random walk converges to Brownian motion.
- Loop-erased random walk converges to SLE(2).

We've learned to interpret these as statements about weak convergence of probability measures. In these particular examples, we can view the discrete objects as continuous curves in a particular metric space.

Random Walk Converges to Brownian Motion

$$\left\{ t \mapsto \frac{1}{n} S(n^2 t \wedge \tau_n) \right\} \xrightarrow{(d)} \{ t \mapsto B(t \wedge \tau_1) \}$$

S – simple random walk on $\mathbb{Z}^2$ with $S_0 = 0$

B – complex Brownian motion with $B_0 = 0$

τ_r – first time curve hits the circle of radius r

Convergence in the strong topology

$$d(\gamma_1, \gamma_2) = |t_{\gamma_1} - t_{\gamma_2}| + \sup_{0 \le t \le t_{\gamma_1} \vee t_{\gamma_2}} |\gamma_1(t) - \gamma_2(t)|$$

where t_γ is the lifetime of the curve γ.

– i.e., weak convergence of probability measures on metric space of curves

– accounts for different random curves running for different lengths of time

Random Walk Converges to Brownian Motion

$$\left\{t \mapsto \frac{1}{n} S(n^2 t \wedge \tau_n)\right\} \xrightarrow{(d)} \{t \mapsto B(t \wedge \tau_1)\}$$

We want convergence of random walk to Brownian motion stopped when it exits the unit disk $\mathbb{D}$. We know (functional CLT) that we need to scale space by the square root of time. It is notationally easier if we scale space by $1/n$; that is, we approximate the disk by

$$\frac{1}{n}\mathbb{Z}^2 \cap \mathbb{D}$$

and so we can equivalently consider random walks on $\mathbb{Z}^2 \cap n\mathbb{D}$. Note that $n^2 \leq \mathbf{E}[\tau_n] \leq (n+1)^2$; we expect the random walk to take $\sim n^2$ steps to exit the ball of radius n. Thus, in order to associate the "correct" continuous curve to the random walk path, we need to introduce the speed function $\sigma_n(t) = \mathbf{E}[\tau_n]t$ or $\sigma_n(t) = n^2 t$.

Important. Not only are the random walk and the Brownian motion traces close, they are close in space at roughly the same time.

Introduction to SLE

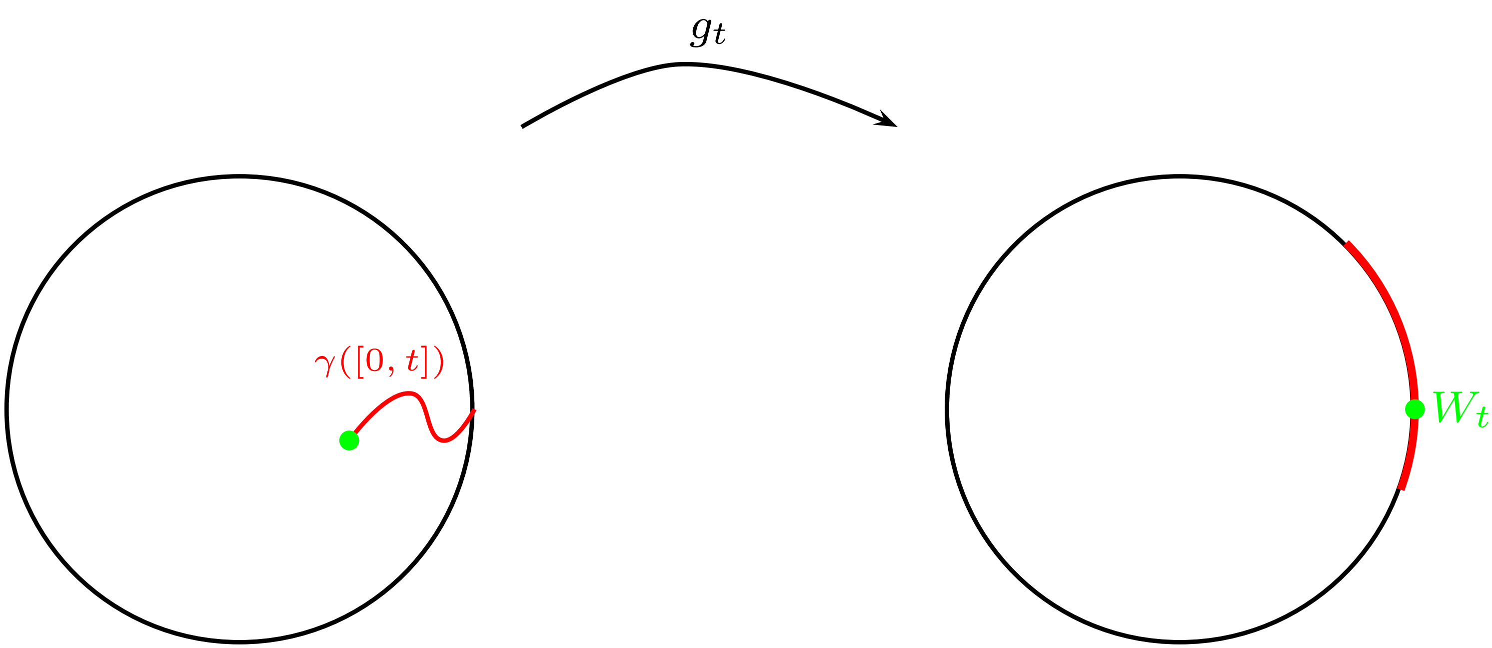
The Schramm-Loewner evolution (SLE) with parameter κ was introduced in 1999 by Oded Schramm while considering possible scaling limits of loop-erased random walk.

Since then, it has successfully been used to study various other lattice models from two-dimensional statistical mechanics including percolation, uniform spanning trees, self-avoiding walk, and the Ising model.

Crudely, one defines a discrete interface on the $1/n$-scale lattice and then lets $n \to \infty$. The limiting continuous "interface" is an SLE.

In "Conformal invariance of planar loop-erased random walks and uniform spanning trees" (AOP 2004), Lawler, Schramm, and Werner showed that the scaling limit of loop-erased random walk is SLE with parameter $\kappa = 2$.

Review of Radial SLE



Reparametrize γ so that

$$g_t'(0) = e^t.$$

This is the capacity parametrization.

Review of Radial SLE (cont)

The evolution of the curve $\gamma(t)$, or more precisely, the evolution of the conformal transformations $g_t : \mathbb{D}_t \to \mathbb{D}$, can be described by the Loewner equation.

For $z \in \mathbb{D}$ with $z \notin \gamma[0, \infty]$, the conformal transformations $\{g_t(z), t \geq 0\}$ satisfy

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z,$$

where

$$W_t = \lim_{z \to \gamma(t)} g_t(z).$$

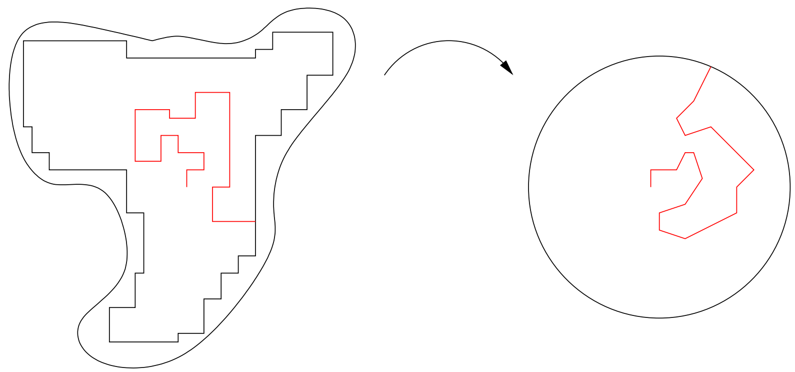
We call W the driving function of the curve γ.

The **radial Schramm-Loewner evolution with parameter $\kappa \geq 0$ with the standard parametrization** is the random collection of conformal maps $\{g_t,\, t \geq 0\}$ obtained by solving the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{e^{i\sqrt{\kappa}B_t} + g_t(z)}{e^{i\sqrt{\kappa}B_t} - g_t(z)}, \quad g_0(z) = z. \tag{LE}$$

where B_t is a standard one-dimensional Brownian motion.

From LERW to SLE



- Let $D \ni 0$ be a simply connected planar domain with $\frac{1}{n}\mathbb{Z}^2$ grid domain approximation $D_n \subset \mathbb{C}$. A grid domain is a domain whose boundary is a union of edges of the scaled lattice. That is, D_n is the connected component containing 0 in the complement of the closed faces of $n^{-1}\mathbb{Z}^2$ intersecting ∂D.
- $\psi_{D_n} : D_n \to \mathbb{D}$, $\psi_{D_n}(0) = 0, \psi'_{D_n}(0) > 0$.
- γ_n: time-reversed LERW from 0 to ∂D_n (on $\frac{1}{n}\mathbb{Z}^2$).
- $\hat{\gamma}_n = \psi_{D_n}(\gamma_n)$ is a path in $\mathbb{D}$. Parameterize by capacity.
- $W_n(t) = W_0 e^{i\vartheta_n(t)}$: the Loewner driving function for $\hat{\gamma}_n$.

Loop-Erased Random Walk Converges to SLE(2)

Consider the following metric on the space of curves in $\mathbb{C}$:

$$\rho(\gamma_1, \gamma_2) = \inf_{\phi} \sup_{0 \le t \le 1} |\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)|$$

where the infimum is over all choices of parametrizations $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $[0, 1]$ of γ_1 and γ_2.

Let μ_n denote the law of γ_n, time-reversed LERW from 0 to ∂D_n, and let μ denote the law of the image in D of radial SLE(2).

Theorem. (Lawler-Schramm-Werner)
The measures μ_n converge weakly to μ as $n \to \infty$ with respect to the metric ρ on the space of curves.

Important. This theorem tells us that the LERW and SLE(2) traces are close. It does not tell us that they are close in space at roughly the same time.

Our Goal

Suppose that X is a LERW on $\mathbb{Z}^2$ started at the origin. We would like

(i) to show that there is a speed function $t \mapsto \sigma_n(t)$ so that

$$t \mapsto \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

converges in law under the strong topology, and

(ii) to identify the limiting curve as SLE(2) in the natural time parametrization that was recently introduced by Lawler-Sheffield and Lawler-Zhou.

Outline

- To discuss a strategy for (i) proving that the limit exists.
- To discuss a strategy for (ii) identifying the limit.
- We'll see how to choose the speed function $\sigma_n(t)$ to execute both strategies

Strategy for (i) Proving that the Limit Exists

Prove tightness!

There are a number of techniques for proving tightness of a stochastic process.

But. . . most of them were designed for Markov processes.

So we'll move to a different setting using an occupation measure.

An Occupation Measure

If γ is a curve, then its occupation measure ν_γ identifies the amount of time γ spends in each Borel subset of $\mathbb{C}$.

Formally,

$$\nu_\gamma(A) := \int_0^{t_\gamma} 1\{\gamma(s) \in A\}\, \mathrm{d}s$$

where A is a Borel subset of $\mathbb{C}$.

Note. Implicit in the statement that γ is a curve is its time parametrization.

- ν_γ is supported on γ
- The total mass of ν_γ is t_γ

Key observation.

occupation measure + curve modulo reparametrization $\Rightarrow$ original curve

An Occupation Measure

Ω – space of continuous curves

$\tilde{\Omega}$ – equivalence classes of curves modulo reparametrization

$\tilde{\Omega} := \Omega/\sim$ where $\gamma_1 \sim \gamma_2$ if $\rho(\gamma_1, \gamma_2) := \inf_{\phi} \sup_{0 \le t \le t_{\gamma_1}} |\gamma_1(t) - \gamma_2(\phi(t))| = 0$

$\tilde{\gamma}$ – the equivalence class $[\tilde{\gamma}]$, the class of curves equivalent to γ wrt ρ.

$\mathcal{M}$ – space of positive Borel measures on $\mathbb{C}$.

Define $T : \Omega \to \tilde{\Omega} \times \mathcal{M}$ by

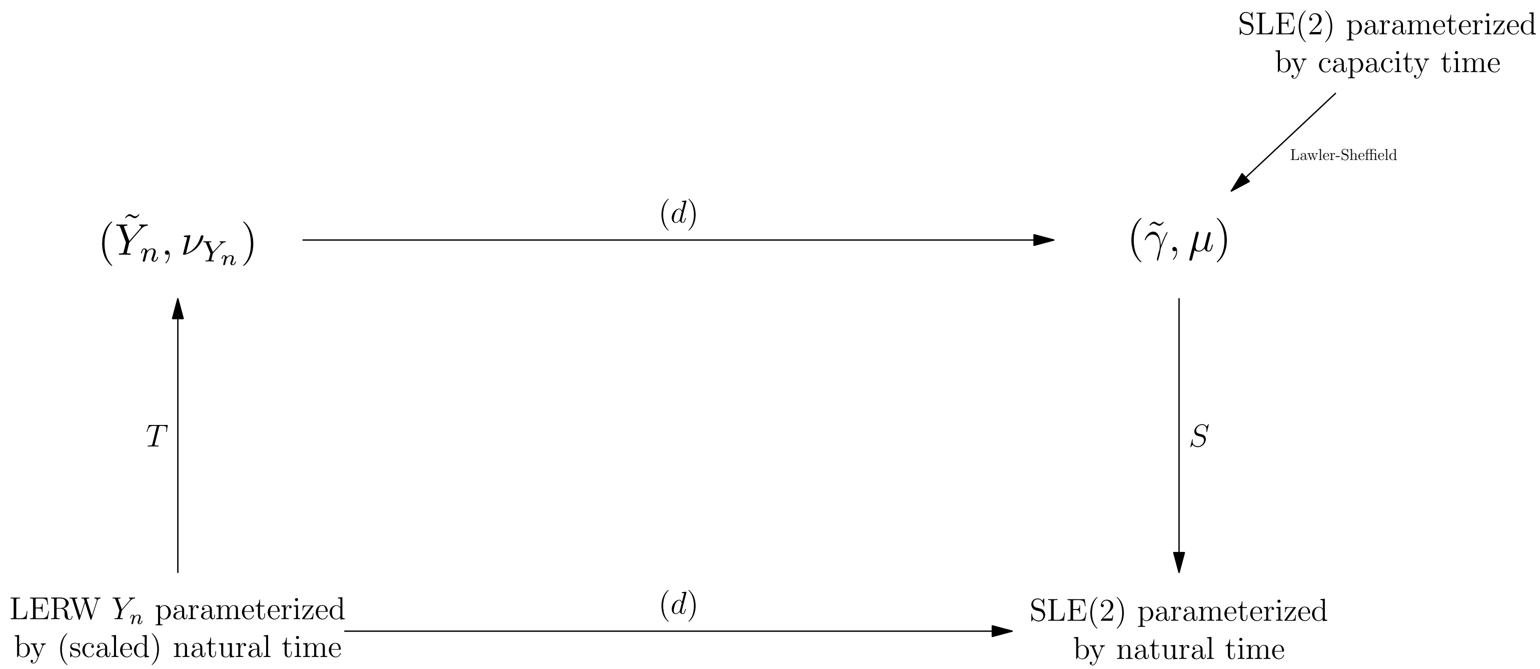
$$T\gamma = (\tilde{\gamma}, \nu_\gamma).$$

Key observation. We can recover γ from the pair $(\tilde{\gamma}, \nu_\gamma)$. Here's how.

If η is <u>any</u> representation of $\tilde{\gamma}$ and $\Theta_\eta(t) = \nu_\gamma(\eta[0,t])$, then

$$\gamma(t) = \eta(\Theta_\eta^{-1}(t)).$$

Hence $(\tilde{\gamma}, \nu_\gamma)$ encodes γ !!!

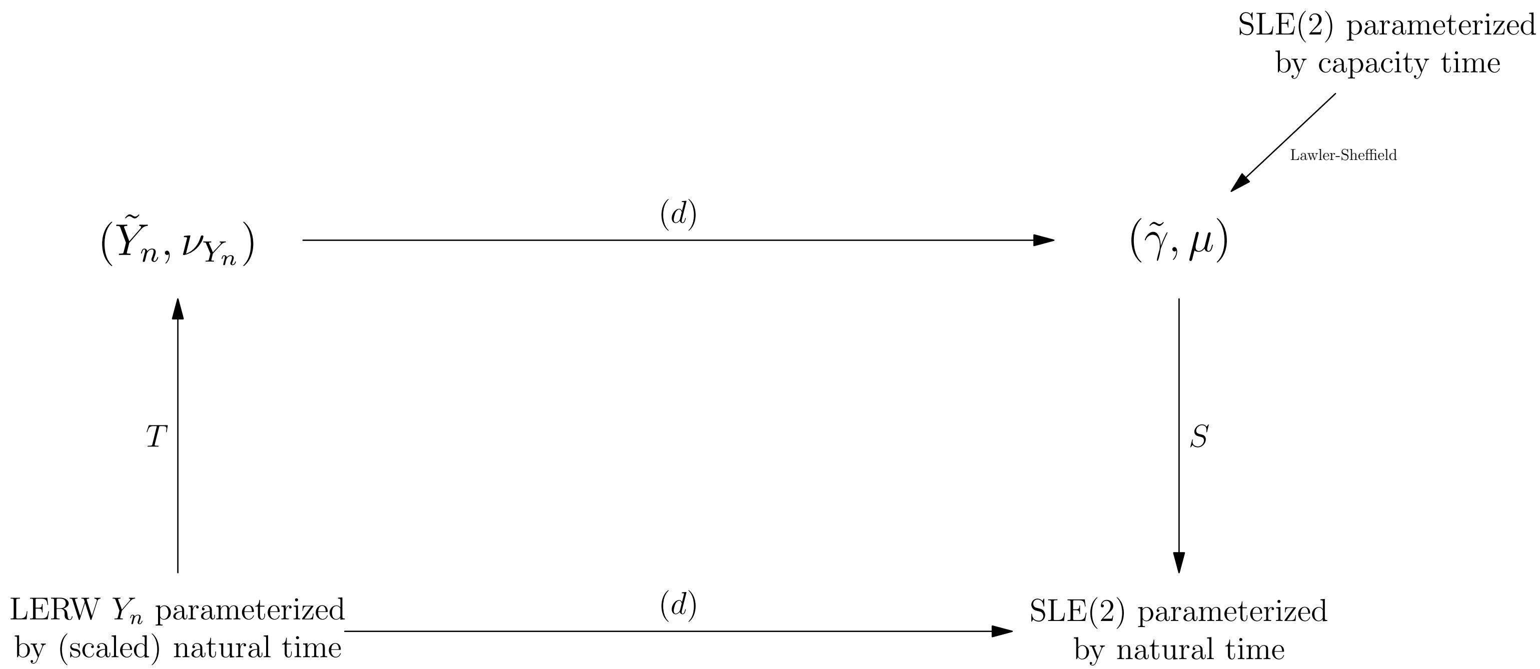


$$Y_n = \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

The topology on the top is the product topology: the one induced by ρ on $\tilde{\Omega}$ along with weak convergence on $\mathcal{M}$.

Convergence on top implies convergence on bottom if T and S are continuous.

T is actually Lipshitz, but S is not continuous (or even well-defined) but it is at all the limit points we will encounter.



$$Y_n = \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Strategy: prove tightness for $(\tilde{Y}_n, \nu_{Y_n})$, then prove uniqueness of subsequential limits.

Advantage: the $\tilde{Y}_n \to \tilde{\gamma}$ part has already been done! (LSW)

For tightness of ν_{Y_n}, it is sufficient to prove that the lifetimes of Y_n are tight.

Consequence: Loop-Erased Random Walk Converges to SLE(2)

Let γ be a radial SLE(2) started uniformly on $\partial\mathbb{D}$.

Suppose that $X(t)$, $0 \le t \le M_n$, denotes the time reversal of a loop-erased random walk on $\mathbb{Z}^2$ started at the origin and stopped at M_n, the time the loop-erased random walk reaches the circle of radius n.

If $z \in \mathbb{D}$, $\epsilon > 0$, and

$$Y_n(t) = \frac{1}{n} X(\sigma_n(t))$$

where $\sigma_n(t)$ is a speed function, then

$$\lim_{n\to\infty} \mathbb{P}\left(\tilde{Y}_n \cap B(z;\epsilon) \neq \emptyset\right) = \mathbb{P}\left(\tilde{\gamma} \cap B(z;\epsilon) \neq \emptyset\right).$$

Strategy for (ii) Identifying the Limit

If γ is SLE in the natural time parametrization, then $(\tilde{\gamma}, \nu_\gamma)$ has certain natural properties, namely it satisfies conformal convariance and the domain Markov property.

In fact, $(\tilde{\gamma}, \nu_\gamma)$ is the unique pair having both properties.

Given tightness, the strategy is to show that all subsequential limits have these properties.

It mimics the original LSW proof.

Conformal Covariance

In other domains, the SLE occupation measure is defined by the conformal covariance rule

$$\nu_{\phi(\gamma)}(\phi(A)) = \int_A |\phi'(z)|^d \nu_\phi(\mathrm{d}z).$$

where d is the Hausdorff dimension of the geometric object in question.

Domain Markov Property

Conditional on some initial segment $\gamma[0,t]$, the measure ν_t^* on $\mathbb{D}$ given by

$$\nu_t^*(g_t(A)) = \int_A |g_t'(z)|^d \nu_\gamma(\mathrm{d}z)$$

for $A \subset \mathbb{D} \setminus \gamma[0,t]$ is independent of $\gamma[0,t]$ and has same law as ν.

Conformal Covariance and Domain Markov Property

Let γ be a naturally parametrized radial SLE from $z \in \partial D$ to $w \in D$.

If $\phi : D \to D'$ is a conformal transformation, then

$$\gamma_t^* = \phi(\gamma_{\varsigma_t})$$

where

$$t = \int_0^{\varsigma_t} |\phi'(\gamma_s)|^d \mathrm{d}s$$

is a naturally parametrized radial SLE in D' from $\phi(z)$ to $\phi(w)$.

Moreover, the law of $s \mapsto \gamma_{t+s}$ conditional on $\gamma[0,t]$ is a naturally parametrized SLE in $D \setminus \gamma[0,t]$ from $\gamma(t)$ to w.

Remark. For chordal SLE, this has been done by G. Lawler, M. Rezaei, S. Sheffield, W. Zhou.

For radial, it has not been rigorously constructed, but is expected to hold.

Properties of $(\tilde{\gamma}, \nu_\gamma)$

(We conjecture that) there is a unique probability measure on $\tilde{\Omega} \times \mathcal{M}$ such that for a pair $(\tilde{\gamma}, \mu)$,

- $\tilde{\gamma}$ is SLE(2) in the unit disk $\mathbb{D}$,
- μ is measurable with respect to $\tilde{\gamma}$,
- if $\gamma \in \tilde{\gamma}$, then $\mu(\cdot \cap \gamma[0,t])$ is measurable wrt $\widetilde{\gamma}[0,t]$,
- $\mathbf{E}[\mathrm{d}\mu(z)] = G(z)\,\mathrm{d}z$ where G is the Green's function for SLE defined by

  $$G(z) = \lim_{\epsilon \to 0+} \epsilon^{-3/4} \mathbf{P}\left\{\gamma \cap B(z;\epsilon) \neq \emptyset\right\},$$

  and
- the domain Markov property holds for μ.

Uniqueness is easy, but existence is hard.

The SLE Green's function

The SLE Green's function is the expected spatial density of the curve.

$$G(z) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbb{P}\left(\mathrm{d}_{\mathrm{conf}}(z, \gamma) \leq \epsilon\right),$$

where $d = 1 + \kappa/8$ is the dimension of the SLE curve and $\mathrm{d}_{\mathrm{conf}}(z, \gamma)$ is one-half times the conformal radius of z from γ.

Ideally one would like to substitute the usual Euclidean distance for the conformal radius, but unfortunately it is not yet known that this limit exists in the radial case.

For chordal SLE, the existence of the limit for conformal radius was proved by G. Lawler, while the existence of the limit was proved by G. Lawler and M. Rezaei for Euclidean distance.

For radial SLE and conformal radius the existence of the limit was proved by T. Alberts, G. Lawler, and M.K .

In the case of chordal SLE, an exact formula for the Green's function is known for all values of $\kappa < 8$. For radial SLE from a prescribed boundary point to a prescribed interior point, an exact formula is known only for $\kappa = 4$. For other values of κ, including $\kappa = 2$, the Green's function can be described in terms of an expectation with respect to radial SLE conditioned to go through a point.

For radial SLE started uniformly on $\partial\mathbb{D}$ and targeting the origin, the Green's function is

$$G_{\mathbb{D}}(z) = |z|^{d-2}.$$

For other simply connected domains D containing the origin, the Green's function is defined by the conformal covariance rule

$$G_D(z) = |\phi'(z)|^{2-d} G_{\mathbb{D}}(\phi(z)) = \left|\frac{\phi'(z)}{\phi(z)}\right|^{2-d},$$

where $\phi : D \to \mathbb{D}$ is a conformal transformation with $\phi(0) = 0$. Equivalently, $G_D(z)$ is the Green's function for radial SLE in D started with respect to harmonic measure on ∂D and targeting the origin.

Idea of Uniqueness

Conformal covariance and the domain Markov property uniquely imply how the conditional expected density of the measure changes as the curve grows:

$$\mathbf{E}[\mathrm{d}\mu(z) \mid \mathcal{F}_t] = |g_t'(z)|^{2-d} G(g_t(z))\, \mathrm{d}z.$$

Therefore,

$$\begin{aligned} \mathbf{E}[\mu(A) \mid \mathcal{F}_t] &= \mu(A \cap \gamma[0,t]) + \mathbf{E}[\mu(A \cap \gamma[0,t]^c) \mid \mathcal{F}_t] \\ &= \mu(A \cap \gamma[0,t]) + \int_{A \cap \gamma[0,t]^c} |g_t'(z)|^{2-d} G(g_t(z))\, \mathrm{d}z. \end{aligned}$$

The uniqueness of the Doob-Meyer decomposition implies the uniqueness of $t \mapsto \mu(A \cap \gamma[0,t])$ and hence uniqueness of μ.

Strategy for (ii) Identifying the Limit

Show that all subsequential limits $(\tilde{\gamma}, \mu)$ of $(\tilde{Y}_n, \nu_{Y_n})$ have the properties that

- $\tilde{\gamma}$ is SLE(2) in the unit disk $\mathbb{D}$,
- μ is measurable with respect to $\tilde{\gamma}$,
- if $\gamma \in \tilde{\gamma}$, then $\mu(\cdot \cap \gamma[0,t])$ is measurable wrt $\widetilde{\gamma}[0,t]$,
- $\mathbf{E}[\mathrm{d}\mu(z)] = G(z)\,\mathrm{d}z$ where G is the Green's function for SLE, and
- the domain Markov property holds for μ.

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Most desirable choice is $\sigma_n(t) = n^{5/4} t$

Based on the long-standing conjecture that M_n "grows like" $n^{5/4}$ where M_n is the number of steps in the LERW (i.e., $M_n = \tau_n$)

Very, very difficult to prove! This would imply that

$$\frac{M_n}{n^{5/4}}$$

has a limiting distribution as $n \to \infty$.

Strongest known result is still that

$$\lim_{n\to\infty} \frac{\log M_n}{\log n} = \frac{5}{4}.$$

(Originally proved by Kenyon, later by Masson.)

But we don't even know how to get tightness!

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

Second choice is $\sigma_n(t) = \mathbf{E}[M_n]t$

This implies that the total lifetime of Y_n is $M_n/\mathbf{E}[M_n]$

Barlow and Masson give tightness bounds for this. In fact, they also give exponential tail bounds

$$\mathbf{P}\left\{\alpha^{-1} \leq \frac{M_n}{\mathbf{E}[M_n]} \leq \alpha\right\} \geq 1 - Ce^{-c\alpha^{1/2}}.$$

"Historical" remark: This result is what really motivated the present work.

Another advantage: If this works, then showing convergence for the first choice of speed function reduces to showing that

$$\mathbf{E}[M_n] \sim cn^{5/4}.$$

How should the speed function by chosen?

$$Y_n(t) := \frac{1}{n} X(\sigma_n(t) \wedge \tau_n)$$

So let's use our second choice:

$$\sigma_n(t) = \mathbf{E}[M_n]t.$$

There are five properties that all subsequential limits need to satisfy. The measurability properties seem okay.

But, we still need to show that all subsequential limits satisfy conformal covariance and the domain Markov property.

Let's focus on trying to prove that

$$\mathbf{E}[\mathrm{d}\mu(z)] = G(z)\,\mathrm{d}z$$

for all subsequential limits μ of ν_{Y_n}.

How should the speed function by chosen?

In the special case that $A = B(z;\epsilon) \subset \mathbb{D}$, we believe the simple geometry of A can be helpful. Write

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon))\right] = \mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid \tilde{Y}_n \cap B(z;\epsilon) \neq \emptyset\right] \mathbb{P}\left(\tilde{Y}_n \cap B(z;\epsilon) \neq \emptyset\right).$$

The second term on the right above converges to $\mathbb{P}\left(\tilde{\gamma} \cap B(z;\epsilon) \neq \emptyset\right)$.

For the first term, we roughly expect that the loop-erased walk goes through $B(z;\epsilon)$ as if it were a loop-erased walk in that domain, i.e., it should not be influenced too much by its future or past. This leads to the conjecture that

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid Y_n \cap B(z;\epsilon) \neq \emptyset\right] = \frac{\mathbf{E}\left[M_{\epsilon n}\right]}{\mathbf{E}\left[M_n\right]} + o(1).$$

How should the speed function by chosen?

To complete the convergence to the Green's function via this strategy, we also expect that

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[M_{\epsilon n}\right]}{\mathbf{E}\left[M_n\right]} = \epsilon^{5/4}[1+o(1)]$$

as $\epsilon \to 0$.

Combining with the previous estimates, this will show that

$$\begin{aligned}\lim_{n\to\infty} \mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon))\right] &= \epsilon^{5/4}\mathbb{P}\left(\tilde{\gamma} \cap B(z;\epsilon) \neq \emptyset\right)[1+o(1)] \\ &= \epsilon^2 G(z)[1+o(1)].\end{aligned}$$

How should the speed function by chosen?

Theorem. If $z \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small, then

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid Y_n \cap B(z;\epsilon) \neq \emptyset\right] \leq C \log(1/\epsilon)\epsilon^{5/4}.$$

Conjecture. If $z \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small, then

$$\mathbf{E}\left[\nu_{Y_n}(B(z;\epsilon)) \mid Y_n \cap B(z;\epsilon) \neq \emptyset\right] = \frac{\mathbf{E}[M_{\epsilon n}]}{\mathbf{E}[M_n]} + o(1)$$

as $n \to \infty$.

To prove the conjecture, need a strong separation lemma. This is currently out of reach.

Says that the curve up until it hits the ball of radius ϵ does not too strongly affect how the curve behaves inside the ball of radius ϵ.

How should the speed function by chosen?

Write $\sigma_n(t) = c_n t$.

It is sufficient to prove that

$$\sum_{e \in A_n} \left[\frac{2n^2}{c_n} \mathbb{P}\left(z_e \in \tilde{Y}_n \right) - G(z_e) \right] = o(n^2)$$

where $A_n = A \cap n^{-1}\mathbb{Z}^2$ so that the sum is over all (undirected) edges e of A_n, and z_e is the midpoint of the edge e.

Carrying out this estimate appears to be genuinely difficult. It is a hard problem to describe asymptotics for the probability that loop-erased walk passes through a particular edge, and even harder to show that the limit is the SLE Green's function. There appears to be work in progress by G. Lawler and F. Johansson Viklund in this direction, although it is not yet clear how sharp the asymptotics will be.